Componentwise Asymptotic Stability of Linear Constant Dynamical Systems

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Abstract—**This note deals with a special type of asymptotic stability, namely componentwise asymptotic stability with respect to the vector $\gamma(t)$ (CWASγ) of system S: $\dot{x} = Ax + Bu$, $t \geqslant 0$, where $\gamma(t) > 0$ (componentwise inequality) and $\gamma(t) \to 0$ as $t \to +\infty$. S is CWASγ if for each $t_0 \geqslant 0$ and for each $|x(t_0)| \leqslant \gamma(t_0)$ ($|x(t_0)|$ with the components $|x_i(t_0)|$ the free response of S satisfies $|x(t)| \leqslant \gamma(t)$ for each $t \geqslant t_0$. For $\gamma(t) \triangleq \alpha e^{-\beta t}$, $t \geqslant 0$, with $\alpha > 0$ and $\beta > 0$ (scalar), the CWEAS (E = exponential) may be defined. S is CWASγ (CWEAS) if and only if $\dot{\gamma}(t) \geqslant \bar{A}\gamma(t)$, $t \geqslant 0$ ($\bar{A}\alpha < 0$); $A \triangleq (a_{ij})$ and $\bar{A}$ has the elements a_{ii} and $|a_{ij}|$, $i \neq j$. These results may be used in order to evaluate in a more detailed manner the dynamical behavior of S as well as to stabilize S componentwise by a suitable linear state feedback.**

I. Introduction

Consider the standard linear constant dynamical system

$$S: \quad \dot{x} = Ax + Bu, \qquad t \geqslant 0, \quad x \in R^n, \quad u \in R^m$$

with the initial condition $x(t_0) \triangleq x_0$, where $A \triangleq (a_{ij})$ and B are real matrices of adequate dimensions.

In certain applications the dynamical systems (e.g., those arising in electrical engineering or in biology) have to satisfy, besides the *sine qua non* condition of asymptotic stability, some more subtle conditions, for instance of the form

$$|x_i(t)| \leqslant \alpha_i e^{-\beta t}, \qquad t \geqslant t_0, \quad i = 1, \cdots, n \tag{1}$$

for each $t_0 \geqslant 0$ and for each $|x_i(t_0)| \leqslant \alpha_i e^{-\beta t_0}$, $i = 1, \cdots, n$, where $x_i(t)$ are the state components of the free or impulse response of S and $\alpha_i > 0$, $\beta > 0$ have prescribed values.

Moreover, in engineering design, one may consider (1) *ab initio* as a performance specification, when the state variables are physically different and/or of different importance for the normal process evolution. Obviously, it depends on the matrix A whether such a performance specification can or cannot be satisfied.

The purpose of this note is to define a special type of asymptotic stability of S, namely the componentwise asymptotic stability [characterization of the form (1)] and to prove some results which may be useful for a more subtle evaluation of the free or impulse response of S as well as for the solution of the problem of componentwise stabilization of S.

II. Main Results

In order to formulate lapidarily our results we begin with some basic notations. Let $v \triangleq (v_i)$ and $w \triangleq (w_i)$ be two vectors in R^n and let $M \triangleq (m_{ij})$ be a $(n \times n)$ real matrix. We denote by $|v|$ the vector with the components $|v_i|$ and by $\bar{M}$ the matrix with the elements m_{ii} and $|m_{ij}|$, $i \neq j$. We also denote by $v > w$ ($v \geqslant w$) or $w < v$ ($w \leqslant v$) and by $M \geqslant 0$ to signify $v_i > w_i$ ($v_i \geqslant w_i$) and $m_{ij} \geqslant 0$ for all i, j.

A. Componentwise Asymptotic Stability

Consider γ: $[0, +\infty) \to R^n$ with the properties: $\gamma(t) > 0$ for $t \geqslant 0$, $\gamma(t)$ is differentiable and

$$\lim_{t \to \infty} \gamma(t) = 0. \tag{2}$$

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Definition 1: The system S is called componentwise asymptotically stable with respect to $\gamma(t)$ (CWASγ) if for each $t_0 \geqslant 0$ and for each $|x_0| \leqslant \gamma(t_0)$ the free response of S satisfies

$$|x(t)| \leqslant \gamma(t) \qquad \text{for each } t \geqslant t_0. \tag{3}$$

Remark 1: Obviously, under condition (2), the inequality (3) holds only if S is asymptotically stable.

Remark 2: Inequality (3) does not evidence only a bound of the absolute value of $x(t)$, which can be determined for each free response of the asymptotically stable S. It is very important to remark that CWASγ actually belongs to the sphere of stability notion because (3), under (2), must hold for each $t_0 \geqslant 0$ and for each $|x_0| \leqslant \gamma(t_0)$.

Remark 3: According to [1]–[6] it follows that CWASγ is equivalent to the *flow invariance* of the time-dependent interval

$$I(t) \triangleq \{v \in R^n;\ |v| \leqslant \gamma(t)\}, \qquad t \geqslant 0$$

for the free response of S on $[t_0, +\infty)$ for each $t_0 \geqslant 0$, i.e., on $[0, +\infty)$. It is known [4] that $I(t)$ is flow invariant for the free response of S on $[0, +\infty)$ if and only if

$$\lim_{h \searrow 0} \frac{1}{h} d(z + hAz;\ I(t+h)) = 0$$
$$\text{for each } t \geqslant 0 \text{ and for each } z \in I(t)$$

where $d(v; I) \triangleq \inf d(v; w)$ for $w \in I$ is the distance from $v \in R^n$ to $I \subset R^n$.

Theorem 1: A necessary and sufficient condition such that S be CWASγ is that

$$\dot{\gamma}(t) \geqslant \bar{A}\gamma(t) \qquad \text{for each } t \geqslant 0. \tag{4}$$

Proof: According to Remark 3, it follows that CWASγ is equivalent to

$$|z + h(Az + a(h))| \leqslant \gamma(t+h) \qquad \text{for each } t \geqslant 0, \text{ for each } z \in I(t),$$
$$\text{for } h > 0, \text{ small enough} \tag{5}$$

and for a certain a: $[0, +\infty) \to R^n$, with $a(h) \to 0$ as $h \searrow 0$. $\gamma(t)$ is differentiable. Then there exists r: $[0, +\infty) \to R^n$, with $r(h) \to 0$ as $h \searrow 0$, such that

$$\gamma(t+h) - \gamma(t) = h\dot{\gamma}(t) + hr(h), \qquad t \geqslant 0.$$

Thus, (5) is equivalent to

$$|z + h(Az + a(h))| \leqslant \gamma(t) + h\dot{\gamma}(t) + hr(h) \qquad \text{for each } t \geqslant 0,$$
$$\text{for each } z \in I(t) \text{ and for } h > 0, \text{ small enough.} \tag{6}$$

Obviously, the vectorial inequality from (6) must also hold for the maximum value and for the minimum value of each component of $z + hAz$ for $h > 0$, small enough, for $t \geqslant 0$ and for $z \in I(t)$. Since $z + hAz$ is linear for z and $I(t)$ has symmetrical limits, the extrema of the ith component of $z + hAz$ for $h > 0$, small enough, can be reached, respectively, for

$$z_{ex}^i = \pm \operatorname{diag}(\operatorname{sgn} a_{i1}, \cdots, \operatorname{sgn} a_{ii-1}, 1, \operatorname{sgn} a_{ii+1}, \cdots, \operatorname{sgn} a_{in})\gamma(t) \in I(t).$$

Thus, for $z = z_{ex}^i$ the ith inequality from (6), after simplification by $h > 0$, is equivalent to

$$a_{ii}\gamma_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\gamma_j(t) \leqslant \dot{\gamma}_i(t) + r_i(h) \mp a_i(h)$$
$$\text{for each } t \geqslant 0, \text{ for } h > 0, \text{ small enough, } i = 1, \cdots, n \tag{7}$$

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where $\gamma_i(t)$, $r_i(h)$, and $a_i(h)$ are the components of $\gamma(t)$, $r(h)$, and $a(h)$, respectively.

Now, taking into account that $a(h) \to 0$ and $r(h) \to 0$ as $h \searrow 0$, the equivalence between (7) and (4) is obvious.

Theorem 2: A necessary and sufficient condition such that S be CWAS γ is that

$$\gamma(t) \geqslant e^{\bar{A}(t-\phi)}\gamma(\phi) \qquad \text{for each } \phi \geqslant 0 \text{ and for each } t \geqslant \phi. \tag{8}$$

Proof: It is known (Bellman [7, p. 172]) that $e^{\bar{A}t} \geqslant 0$ for $t \geqslant 0$. For a continuous function v: $[0, +\infty) \to R^n$, with $v(t) \geqslant 0$ for $t \geqslant 0$ and chosen such that

$$\dot{\gamma}(t) = \bar{A}\gamma(t) + v(t) \qquad \text{for each } t \geqslant 0,$$

(i.e., (4) with "=" in place of "$\geqslant$"), one can deduce

$$\dot{\gamma}(t) = e^{\bar{A}(t-\phi)}\gamma(\phi) + \int_{\phi}^{t} e^{\bar{A}(t-t')}v(t')\, dt'$$

$$\text{for each } \phi \geqslant 0 \text{ and for each } t \geqslant \phi.$$

As $e^{\bar{A}(t-t')}v(t') \geqslant 0$ for $t' \in [\phi, t]$ it follows that (4) is equivalent to (8).

Theorem 3: A necessary and sufficient condition for the existence of $\gamma(t)$ such that S be CWAS γ is that $\bar{A}$ be Hurwitzian.

Proof:

Sufficiency: If $\bar{A}$ is Hurwitzian, then there exists $\gamma(t)$, for instance $\gamma(t) \triangleq e^{\bar{A}t}\gamma(0)$, for which (2) and (8) are satisfied.

Necessity: If (2) and (8) hold, then $\lim_{t\to\infty} e^{\bar{A}t}\gamma(0) \leqslant \lim_{t\to\infty}\gamma(t) = 0$. Since $e^{\bar{A}t} \geqslant 0$ for $t \geqslant 0$ and $\gamma(0) > 0$ it follows that $\lim_{t\to\infty} e^{\bar{A}t} = 0$, i.e., $\bar{A}$ is Hurwitzian.

Remark 4: Let Γ be the Abelian semigroup of the solutions of (4) with $\bar{A}$ Hurwitzian. Obviously, S is CWAS γ for each $\gamma \in \Gamma$. Moreover, for each pair $\gamma_1, \gamma_2 \in \Gamma$ the CWAS γ_1 is equivalent to CWAS γ_2. This allows us to specialize $\gamma(t)$ and to characterize in a more explicit manner the free response of S.

B. Componentwise Exponential Asymptotic Stability

Consider

$$\gamma(t) \triangleq \alpha e^{-\beta t}, \qquad t \geqslant 0 \tag{9}$$

where $\alpha \triangleq (\alpha_1, \cdots, \alpha_n)$ and $\beta > 0$ (scalar).

Definition 2: The system S is called componentwise exponential asymptotically stable (CWEAS) if there exist $\alpha > 0$ and $\beta > 0$ such that for each $t_0 \geqslant 0$ and for each $|x_0| \leqslant \alpha e^{-\beta t_0}$ the free response of S satisfies

$$|x(t)| \leqslant \alpha e^{-\beta t} \qquad \text{for each } t \geqslant t_0.$$

Theorem 4: A necessary and sufficient condition such that S be CWEAS is that

$$0 < \beta \leqslant \min_i \left(-a_{ii} - \frac{1}{\alpha_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\alpha_j \right). \tag{10}$$

To prove this theorem one has to determine the necessary and sufficient condition (10) such that (9) be a solution of (4).

Theorem 5: A necessary and sufficient condition such that S be CWEAS is that

$$\bar{A}\alpha < 0. \tag{11}$$

The proof follows immediately by the fact that (10) is equivalent to $\bar{A}\alpha \leqslant -\beta\alpha < 0$.

Remark 5: Referring to (11) we may naturally state the problem of the existence of a solution $\alpha > 0$ for inequation (11). We recall that all elements of $\bar{A}$ which do not belong to the first diagonal are nonnegative. If there exists $\alpha > 0$ such that (11) holds, then $-\bar{A}$ is an M-matrix (Ostrowski, [7, p. 295]). An equivalent definition of an M-matrix is that all its principal minors be positive. Under these circumstances one may formulate the following.

Theorem 6: A necessary and sufficient condition such that S be CWEAS is that

$$(-1)^k \bar{A}_k > 0, \qquad k = 1, \cdots, n \tag{12}$$

where $\bar{A}_k$ are the principal minors of $\bar{A}$.

Certainly, the proof follows by the fact that S is CWEAS if and only if $-\bar{A}$ is an M-matrix. A direct proof can be also given by using the Gaussian elimination process, [8].

Another similar result [6] may be established by using the theorem of the existence of a solution α for the inequation

$$\begin{bmatrix} \bar{A} \\ -I \end{bmatrix} \alpha < 0 \qquad (I \text{ is the unit } (n \times n) \text{ matrix})$$

due to Dines [9].

C. Dependence on Vector Basis

Remark 6: The CWAS γ implies the asymptotic stability in the sense of Lyapunov. Consequently, each of Theorems 1–6 is a criterion for asymptotic stability [6]. Thus, a useful result may be the following.

Theorem 7: The system S is asymptotically stable if (12) holds.

Proof: Equation (12) implies (11). In the sequel we give a direct proof, which is interesting by itself. One can equivalently express (11) as

$$-a_{ii} > \frac{1}{\alpha_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\alpha_j, \qquad i = 1, \cdots, n. \tag{13}$$

Consider the matrix

$$A_\alpha = \operatorname{diag}(\alpha_1^{-1}, \cdots, \alpha_n^{-1})\, A \operatorname{diag}(\alpha_1, \cdots, \alpha_n)$$

and note that A_α is similar to A, both having the same spectrum $\sigma(A)$. The α-Gershgorin's disks associated to A, i.e., the Gershgorin's disks associated to A_α [7, p. 106] are the subsets

$$G_i(A_\alpha) \triangleq \left\{ s \in C;\ |s - a_{ii}| \leqslant \frac{1}{\alpha_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\alpha_j \right\}, \qquad i = 1, \cdots, n$$

which have the remarkable property $\sigma(A) \subset \cup_{i=1}^{n} G_i(A_\alpha)$. According to (13) it follows that $\sigma(A) \subset \{s \in C;\ \operatorname{Re} s < 0\}$, i.e., S is asymptotically stable.

Remark 7: The asymptotic stability does not imply CWAS γ because the latter depends on the particular choice of the state vector of S. To illustrate this let us consider

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$$

with $a_1 > 0$, $a_2 > 0$ (A is Hurwitzian). It is easy to see that $\bar{A}$ is not Hurwitzian, i.e., S cannot be CWAS γ.

A natural question is that of the existence of a vector basis in R^n for which an asymptotically stable system is also CWAS γ.

Theorem 8: Consider A Hurwitzian and $\bar{A}$ non-Hurwitzian. There exists at least one transformation $\tilde{x} = Px$, with $\det P \neq 0$, for S such that $\tilde{S}$ be CWAS γ if

$$\sigma(A) \subset \{s \in C; \operatorname{Re} s < 0,\ |\operatorname{Im} s| < -\operatorname{Re} s\}. \tag{14}$$

Proof: Let us consider $P = V_R^{-1}$, where V_R is the modal matrix of A over R. Such being the case $\tilde{A} = V_R^{-1} A V_R$ is the (block) diagonal or the (block) Jordan canonical form of A over R. If (14) is valid, then $\bar{\tilde{A}}$ is Hurwitzian.

III. Example

Consider

$$A = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$$

and determine $\alpha_1 > 0$, $\alpha_2 > 0$, and $\beta > 0$ for which S is CWEAS.

According to (10) one may write

$$0 < \beta \leqslant \min[(1-2(\alpha_2/\alpha_1));\ (3-\alpha_1/\alpha_2)]$$

which leads to $2 < \alpha_1/\alpha_2 < 3$ and, for $0 < \beta < 1$, to $2/(1-\beta) \leqslant 3-\beta$. From the latter one obtains $0 < \beta \leqslant 2-\sqrt{3}$. For $\beta = 2-\sqrt{3}$ one deduces $\alpha_1 = (1+\sqrt{3})\rho$ and $\alpha_2 = \rho$, where $\rho > 0$.

IV. Concluding Remarks

The CWAS γ is a special type of asymptotic stability, depending on the vector basis in R^n. This represents a row property of the evolution matrix A, which holds if and only if $\bar{A}$ is Hurwitzian. Moreover, the CWEAS corresponds to a certain dominance of the first diagonal elements of A in the row direction, necessarily implying, also for CWEAS γ, that these elements are negative [see (13)].

The results concerning CWAS γ and CWEAS are easily applicable and they allow a more subtle evaluation of the dynamical behavior of the linear constant systems by means of the free response (*mutatis mutandis* by means of the impulse or step response). This may be necessary especially when the state components are physically different and/or of different importance for the normal process evolution. Such a case can be, for instance, the dc electric motor whose loading torque is constant except for some moments when impulsive components may be added. If so it is reasonable to impose restrictions regarding the maximum value and the decay speed of the variations of loading current and of angular velocity for arbitrarily located initial conditions (compatible with the restrictions and determined by the impulsive components of loading torque). Another example, this time from biology, was considered by Pavel [10].

If a system is not CWAS γ, then one may naturally state the componentwise stabilization problem, [11], [12]. From the proof of Theorem 7 it follows that its solution consists in the assignment of α-Gershgorin's disks in the half complex plane $\operatorname{Re} s < 0$ by a suitable linear state feedback.

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Stability of an Exponentially Stabilizable System

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***Abstract*—Let A be the generator of a C_0 semigroup $T(t)$, $t \geqslant 0$, and denote by $S(t)$, $t \geqslant 0$, the semigroup generated by $A - K$, where K is a bounded linear operator on a Hilbert space. In this note we find necessary and sufficient conditions for the original semigroup $T(t)$, $t \geqslant 0$, to be exponentially stable, *given* that the "feedback" semigroup $S(t)$, $t \geqslant 0$, is exponentially stable. Applications to feedback stabilization via a steady-state Riccati equation will then be made.**

I. Introduction

Let A be the generator of a strongly continuous (i.e., of the class C_0) semigroup of bounded linear operators $T(t)$, $t \geqslant 0$, on an infinite dimensional Hilbert space H with inner product $[\cdot,\cdot]$ and norm $\|\cdot\|$. Let K be a bounded linear operator on H and denote by $S(t)$, $t \geqslant 0$, the semigroup generated by $A - K$. In [1], Gibson proves the following theorem.

Theorem: If $T(t)$, $t \geqslant 0$, is contractive ($\|T(t)\| \leqslant 1$ for all $t \geqslant 0$), and strongly stable (for each x in H: $\|T(t)x\| \to 0$, $t \to \infty$), the operator K is compact. Then, if $S(t)$, $t \geqslant 0$, is exponentially stable ($\|S(t)\| \leqslant Me^{-\alpha t}$, for some $M \geqslant 1$ and $\alpha > 0$), so is $T(t)$, $t \geqslant 0$.

This interesting result implies that if the linear dynamic system

$$\dot{x} = Ax + Bu,$$

where A is the generator of a contraction semigroup and B is bounded linear from another Hilbert space U(say) to H, is strongly stable but *not* exponentially stable. Then it is not possible to exponentially stabilize it, using a feedback control $u = -Fx$, where the state feedback operator F is compact. In other words, the resulting system

$$\dot{x} = (A - BF)x$$

is, in this case, not exponentially stable.

In this note, in view of the above result, we wish to study the following somewhat general problem: "under what conditions will a C_0 semigroup $T(t)$, $t \geqslant 0$,—with generator A—be exponentially stable, given that the semigroup $S(t)$, $t \geqslant 0$, generated by $A - K$, is exponentially stable?" This is studied in Section II, first for a "general" operator K, then for the important case in which K is "generated" from a steady-state Riccati equation (SSRE). A necessary and sufficient condition for the semigroup $T(t)$, $t \geqslant 0$, to be exponentially stable is obtained in this case. The Gibson result is then reproved for the case of a completely nonunitary contraction semigroup $T(t)$, $t \geqslant 0$, and a compact operator B, assuming that the SSRE admits a nonnegative solution.

II. Main Results

We begin by recalling the following results due to Datko [2] which play a key role in this section.

Theorem 1: For a C_0 semigroup $T(t)$, $t \geqslant 0$, on H, with generator A, the following conditions are equivalent.

i) $T(t)$, $t \geqslant 0$, is exponentially stable.

ii) There exist a self-adjoint positive operator $Q > 0$, and a self-adjoint strictly positive operator W, i.e., $W \geqslant kI$ for some $k > 0$, on H such that $2\operatorname{Re}[QAx, x] = -[Wx, x]$, for x in the domain $\mathscr{D}(A)$ of A.

iii) The integral $\int_0^\infty \|T(t)x\|^2\, dt$ is convergent for every x in H.

Suppose now that the semigroup $S(t)$, $t \geqslant 0$, with generator $A - K$, is exponentially stable. Then by Theorem 1 ii)

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