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FREE RESPONSE CHARACTERIZATION VIA FLOW INVARIANCE

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Abstract. It is shown that the flow invariance can be an efficient tool for a more detailed characterization of the dynamic processes. For the linear constant dynamical systems described by $\dot{x} = Ax$, $t \geq 0$, with $A = (a_{ij})$ a $(n \times n)$ real matrix, and for the time-dependent state interval $I(t) = \{z \in R^n; |z| \leq \gamma(t)\}$, $t \geq 0$, as flow invariant set ($|\cdot|$ and $\leq$ signifying componentwise absolute value and inequality respectively), where $\gamma(t)$ is differentiable, a componentwise characterization is developed. A necessary and sufficient condition such that $|x(t)| \leq \gamma(t)$ for each $t_o \geq 0$, for each $|x(t_o)| \leq \gamma(t_o)$ and for each $t \geq t_o$ is that $\bar{A}\gamma(t) \leq \dot{\gamma}(t)$ for each $t \geq 0$. $\bar{A}$ has the elements a_{ii} and $|a_{ij}|$, $i \neq j$. For $\gamma(t) \to 0$ as $t \to \infty$ the componentwise asymptotic stability is defined. Such being the case a necessary and sufficient condition for the existence of $\gamma(t)$ is that $\bar{A}$ be Hurwitzian. The results of the paper may be also used for solving the componentwise stabilization problem.

Keywords : System theory ; linear systems ; time-domain analysis ; free response componentwise characterization ; stability ; componentwise asymptotic stability.

INTRODUCTION

Let us consider the linear constant dynamical system

$$\dot{x} = Ax, \quad t \geq 0, \quad x \in R^n, \tag{1}$$

where $A = (a_{ij})$, $a_{ij} \in R$, with the initial condition

$$x(t_o) = x_o, \quad t_o \geq 0. \tag{2}$$

It is known that the asymptotic stability of the trivial solution in the sense of Liapunov is conceived on the basis of the norm in R^n. This means that the temporal evolution of the solution of the Cauchy problem (1),(2), which has the form

$$x(t) = e^{A(t - t_o)}x_o, \quad t \geq t_o, \tag{3}$$

is evaluated via the scalar function $\|x(t)\|$, $t \geq t_o$. A well-known result in this respect is the following : The system (1) is asymptotically stable if and only if there exist $M > 0$, $\beta > 0$ such that for each $t_o \geq 0$, for each $x_o \in R^n$ and for each $t \geq t_o$

$$\|x(t)\| \leq M\|x_o\|e^{-\beta(t - t_o)} \tag{4}$$

holds.

In certain theoretical problems as the flow invariance of the solution (3) with respect to a compact subset $I \subset R^n$ or in certain applications, especially in system project and engineering, a more detailed characterization for the temporal evolution of system (1) is desirable. An example of such characterization may be that certain components, or all, of (3), satisfy inequalities of the form (4).

The purpose of this paper is to develop a componentwise characterization for the free response (3) of the system (1) and subsequently to state the componentwise asymptotic stability problem. In this respect we prove some simple and easily applicable results, by using flow invariance methods (Crandall, 1972; Martin, 1973; Nagumo, 1942; Pavel and Vrabie, 1979; Pavel, 1982).

FLOW INVARIANCE OF A TIME-DEPENDENT INTERVAL

We begin with some basic notations and definitions. Let $v=(v_i)$, $w=(w_i)$ be two vectors in R^n and let $C=(c_{ij})$, $D=(d_{ij})$ be two real $(n \times n)$ matrices. In all what follows we denote by $|v|$ the vector with the components $|v_i|$, by $|C|$ the matrix with the elements $|c_{ij}|$ and by $\bar{C}$ the matrix with the elements c_{ii} and $|c_{ij}|$, $i \neq j$. We also denote by $v > w$ ($v \geq w$) and by $C > D$ ($C \geq D$) to signify $v_i > w_i$ ($v_i \geq w_i$) and $c_{ij} > d_{ij}$ ($c_{ij} \geq d_{ij}$) for all i,j. Let $\gamma_i(t) > 0$, $t \geq 0$, $i=1,..,n$, be n differentiable scalar functions, which define on each unit vector basis of R^n the vector function $\gamma(t) > 0$, $t \geq 0$, with the components $\gamma_i(t)$, and the time-dependent state interval

$$I(t) = \{v \in R^n;\ \Upsilon(t) - |v| \geq 0\},\quad t \geq 0. \quad (5)$$

For the free response (3) of (1), which may belong to I(t) on a certain time interval, the following definitions concerning the flow invariance are available.

Definition 1. I(t) is flow invariant for (1) if for each $t_o \geq 0$ and for each $x_o \in I(t_o)$ there exists $t_f > 0$ such that the free response (3) satisfies

$$x(t) \in I(t) \text{ for each } x_o \in I(t_o) \text{ and for each } t \in [t_o, t_o + t_f]. \quad (6)$$

Definition 2. I(t) is globally flow invariant for (1) if for each $t_o \geq 0$ the free response (3) satisfies

$$x(t) \in I(t) \text{ for each } x_o \in I(t_o) \text{ and for each } t \geq t_o. \quad (7)$$

As for each $t \geq 0$, I(t) is compact and for each $t_o \geq 0$ the free response (3) of (1) is defined and continous on $[t_o, +\infty)$, one may prove the following.

Proposition 1[1]. A necessary and sufficient condition such that I(t) be flow invariant for (1) is that I(t) be globally flow invariant for (1).

Proof. Sufficiency is obvious. Necessity. Let us consider $T = \{t \in (t_o + t_f, +\infty);\ x(t) \notin I(t)\}$ and $\mathfrak{t} = \inf T$. Clearly, (7) holds if and only if $T = \emptyset$ ($\mathfrak{t} = +\infty$). Suppose by contradiction that $\mathfrak{t} < +\infty$. If $\mathfrak{t} \in T$, then $x(\mathfrak{t}) \notin I(\mathfrak{t})$. But $x(t) \in I(t)$ for each $t \in [t_o, \mathfrak{t})$ and therefore for $t \nearrow \mathfrak{t}$, $x(t) \to x(\mathfrak{t}) \in I(\mathfrak{t})$, what is impossible. If $\mathfrak{t} \notin T$, then $x(\mathfrak{t}) \in I(\mathfrak{t})$. But $\mathfrak{t}$ cannot be an isolated point of flow invariance, what implies that $\mathfrak{t} \neq \inf T$. This contradicts the definition of $\mathfrak{t}$. Hence $\mathfrak{t} = +\infty$ and $T = \emptyset$.

Remark 1. According to Proposition 1 the initial time $t_o \geq 0$ and the length $t_f > 0$ of flow invariance of I(t) for (1) can be arbitrary. Therefore, from the basic result of Pavel and Vrabie (1979) we may derive the following.

Theorem 1. A necessary and sufficient condition such that I(t) be globally flow invariant for (1) is that

$$\lim_{h \searrow 0} h^{-1} d(z + hAz;\ I(t+h)) = 0 \quad \text{for each } t \geq 0 \text{ and for each } z \in I(t). \quad (8)$$

We denoted in (8) $d(v;I) = \inf \|v - w\|$ for $w \in I$.

Using Theorem 1 explicitely for (1) and I(t), we can establish the following more detailed result, which is useful for analytic and computational purposes.

Theorem 2. A necessary and sufficient condition such that I(t) be globally flow invariant is that

$$\dot{\Upsilon}(t) \geq \bar{A}\Upsilon(t) \text{ for each } t \geq 0. \quad (9)$$

Proof. It is known that (8) is equivalent to

$$|z + h(Az + a(h))| \leq \Upsilon(t+h) \text{ for each } t \geq 0, \text{ for each } z \in I(t), \text{ for } h > 0, \text{ small enough}, \quad (10)$$

and for certain $a : [0, +\infty) \to R^n$, with $a(h) \to 0$ as $h \searrow 0$ (Pavel, 1982). As $\Upsilon(t)$ is differentiable, there exists $r : [0, +\infty) \to R^n$, with $r(h) \to 0$ as $h \searrow 0$ such that $\Upsilon(t+h) - \Upsilon(t) = h\dot{\Upsilon}(t) + hr(h)$, $t \geq 0$. In view of (10) the statement (8) is equivalent to

$$|z + h(Az + a(h))| \leq \Upsilon(t) + h\dot{\Upsilon}(t) + hr(h) \text{ for each } t \geq 0, \text{ for each } z \in I(t) \text{ and } h > 0, \text{ small enough}. \quad (11)$$

Substituting z successively by $\pm(\Upsilon_1, g_{12}, \ldots, g_{1n}), \ldots, \pm(g_{i1}, \ldots, g_{ii-1}, \Upsilon_i, g_{ii+1}, \ldots, g_{in}), \ldots, \pm(g_{n1}, \ldots, g_{nn-1}, \Upsilon_n)$, where $\Upsilon_i = \Upsilon_i(t)$ and $g_{ij} = \Upsilon_j(t)\,\mathrm{sgn}\, a_{ij}$, $i, j = 1, \ldots, n$, $j \neq i$, we ascertain that each row of $(z + hAz)$ reaches its maximum value (for +), respectively its minimum value (for -) and simultaneously each row of (11) can be simplified by $h > 0$ for fixed $t \geq 0$, for $z \in I(t)$ and for fixed $h > 0$, small enough. Consequently (11) is equivalent to $\bar{A}\Upsilon(t) \leq \dot{\Upsilon}(t) + r(h) \mp a(h)$ for each $t \geq 0$. In view of the fact that $a(h) \to 0$ and $r(h) \to 0$ as $h \searrow 0$, it follows that (8) is equivalent to (9).

For a further characterization of the behaviour of (3) regarding this special case of flow invariance, let us consider the system

$$\dot{y} = \bar{A}y,\quad t \geq 0,\quad y \in R^n, \quad (12)$$

with the initial condition

$$y(t_o) = y_o,\quad t_o \geq 0. \quad (13)$$

The solution of (12),(13) is

$$y(t) = e^{\bar{A}(t - t_o)} y_o,\quad t \geq t_o. \quad (14)$$

Notice that the elements of $\bar{A}$, which do not belong to the first diagonal are nonnegative. Via a classical result (Bellman, 1960) it follows that (14) satisfies

$$e^{\bar{A}(t - t_o)} y_o \geq 0, \text{ for each } t_o \geq 0, \text{ for each } y_o \geq 0 \text{ and for each } t \geq t_o. \quad (15)$$

Theorem 3. A necessary and sufficient condition such that I(t) be globally flow invariant for (1) is that

$$\Upsilon(t) \geq e^{\bar{A}(t - \theta)}\, \Upsilon(\theta) \quad \text{for each pair } \theta, t \in [0, +\infty),\ t \geq \theta. \quad (16)$$

[1] Proved by I. Vrabie.

Proof. Sufficiency. Substituting $e^{\bar{A}(t-\theta)}$ by its Taylor expansion around the point $t=\theta$, (16) may be brought to the form $[\Upsilon(t)-\Upsilon(\theta)]/(t-\theta)\geq\bar{A}\Upsilon(\theta)+\bar{A}^2(t-\theta)\Upsilon(\theta)/2!+ +\ldots$, $t>\theta$. Obviously for $\theta\nearrow t$ one obtains (9). Necessity. For each continously function $v:[0,+\infty)\to R^n$, with $v(t)\geq 0$ and such that $\dot{\Upsilon}(t)=\bar{A}\Upsilon(t)+v(t)$ for each $t\geq 0$ (i.e. (9) with "=" in place of "$\geq$"), one deduces

$$\Upsilon(t) = e^{\bar{A}(t-\theta)}\Upsilon(\theta) + \int_\theta^t e^{\bar{A}(t-t')}v(t')dt'$$

for each pair $\theta, t\in[0,+\infty)$, $t\geq\theta$. According to (15), it follows that (9) implies (16).

Remark 2. Taking $t_o=0$ and substituting y_o in (15) successively by $(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$ one concludes that

$$e^{\bar{A}t} \geq 0 \text{ for each } t\geq 0. \quad (17)$$

Corollary 3.1. Each real (nxn) matrix A satisfies

$$|e^{At}| \leq e^{\bar{A}t} \text{ for each } t\geq 0. \quad (18)$$

Proof. Let $\Upsilon(t)=e^{\bar{A}t}\Upsilon(0)$, $t\geq 0$, which satisfies (16) for each $\Upsilon(0)>0$. In view of (3) and (7), for $t_o=0$, one may write $|x(t)|=|e^{At}x_o|\leq e^{\bar{A}t}\Upsilon(0)$ for each $\Upsilon(0)>0$, for each $|x_o|\leq\Upsilon(0)$ and for each $t\geq 0$. Clearly, for $x_o=\pm\Upsilon(0)$ one obtains (18), because $\Upsilon(0)>0$ is arbitrary.

COMPONENTWISE ASYMPTOTIC STABILITY

To define the componentwise asymptotic stability via the flow invariance it is natural to suppose that $\Upsilon_i(t)$, $i=1,\ldots,n$, have also the property

$$\lim_{t\to\infty}\Upsilon_i(t) = 0, \quad i=1,\ldots,n. \quad (19)$$

Definition 3. The system (1) is called componentwise asymptotically stable with respect to $\Upsilon(t)$, under (19), if for each $t_o\geq 0$ and for each $|x_o|\leq\Upsilon(t_o)$ the free response (3) satisfies

$$|x(t)|\leq\Upsilon(t) \text{ for each } t\geq t_o. \quad (20)$$

Remark 3. The componentwise asymptotic stability with respect to $\Upsilon(t)$ is equivalent to the globally flow invariance of $I(t)$ (given by (5) with (19)) for (1). According to Theorems 2,3 one can formulate the following.

Theorem 4. A necessary and sufficient condition such that (1) be componentwise asymptotically stable with respect to $\Upsilon(t)$ is (9) with (19).

Theorem 5. A necessary and sufficient condition such that (1) be componentwise asymptotically stable with respect to $\Upsilon(t)$ is (16) with (19).

Remark 4. One may equivalently restate (9) as

$$\dot{\Upsilon}_i(t) \geq a_{ii}\Upsilon_i(t) + \sum_j |a_{ij}|\Upsilon_j(t) \quad (21)$$
$$\text{for each } t\geq 0, \quad i=1,\ldots,n,$$

where $\sum_j$ signifies the sum for $j=1,2,\ldots,i-1,i+1,\ldots,n$. According to (19), it follows that for each $i=1,\ldots,n$, there exists $t_i\geq 0$, $i=1,\ldots,n$, such that $\dot{\Upsilon}_i(t_i)<0$, $i=1,\ldots,n$. In view of $\Upsilon_i(t_i)>0$, $i=1,\ldots,n$, from (21) it follows that

$$a_{ii}<0, \quad i=1,\ldots,n. \quad (22)$$

Thus, we proved the following.

Theorem 6. A necessary condition such that (1) be componentwise asymptotically stable with respect to $\Upsilon(t)$ is (22).

From (16), under (19), one can easily derive an existence condition for $\Upsilon(t)$.

Theorem 7. A necessary and sufficient condition for the existence of $\Upsilon(t)$ such that (1) be componentwise asymptotically stable with respect to $\Upsilon(t)$ is that $\bar{A}$ be Hurwitzian.

Proof. Sufficiency. If $\bar{A}$ is Hurwitzian, then one can take $\Upsilon(t)=e^{\bar{A}t}\Upsilon(0)$, for which (16) and (19) are satisfied. Necessity. If (1) is componentwise asymptotically stable, then (16) and (19) hold. Suppose by contradiction that $\bar{A}$ is not Hurwitzian, i.e. $\limsup_{t\to\infty}\|e^{\bar{A}t}\|>0$. Taking $\theta=0$ and using (17) and the inequality $\Upsilon(t)>0$, from (16) it follows that

$$\lim_{t\to\infty}\|\Upsilon(t)\|\geq\limsup_{t\to\infty}\|e^{\bar{A}t}\Upsilon(0)\|>0,$$

which contradicts (19).

COMPONENTWISE EXPONENTIAL ASYMPTOTIC STABILITY

By specializing the functions $\Upsilon_i(t)$, i.e.

$$\Upsilon_i(t) = \alpha_i e^{-\beta t}, \quad t\geq 0, i=1,\ldots,n, \quad (23)$$

where $\alpha_i>0$, $i=1,\ldots,n$, are the components of the vector $\alpha=(\alpha_1,\ldots,\alpha_n)>0$ and $\beta>0$ is a scalar, one may develop a more explicite characterization for (3).

Definition 4. The system (1) is called componentwise exponential asymptotically stable if there exist $\alpha>0$ and $\beta>0$ such that for each $t_o\geq 0$ and for each $|x_o|\leq\alpha e^{-\beta t_o}$ the free response (3) satisfies

$$|x(t)|\leq\alpha e^{-\beta t} \text{ for each } t\geq t_o. \quad (24)$$

Proposition 2. A necessary and sufficient condition such that (1) be componentwise asymptotically stable with respect to $\Upsilon(t)$ is that (1) be componentwise exponential asymptotically stable.

Proof. Sufficiency is obvious. Necessity. According to the theorem of Perron - Frobenius (Bellman, 1960), there is for each $t\geq 0$ an unique eigenvalue $e^{-rt}>0$ of $e^{\bar{A}t}\geq 0$ (see (17)), with $r>0$ ($\bar{A}$ is Hurwitzian), which has greatest absolute value. This eigenvalue is simple and its associated eigenvector may be taken $v>0$. Thus, one can write $e^{\bar{A}t}v=e^{-rt}v, t\geq 0$. Integrating this relation on $[0,+\infty)$ and then multiplying by $e^{-\beta t}$, with $0<\beta\leq r$, one obtains easily that $d(e^{-\beta t}v)/dt \geq \bar{A}(e^{-\beta t}v)$ for each $t\geq 0$. This means that (9) is satisfied for $\Upsilon(t)$ given by (23) with $\alpha=v$, i.e. (24) holds.

In this context, according to Theorem 4, one may prove.

Theorem 8. A necessary and sufficient condition such that (1) be componentwise exponential asymptotically stable is that

$$0<\beta\leq \min_i(-a_{ii} - \alpha_i^{-1}\sum_j |a_{ij}|\alpha_j). \quad (25)$$

The proof follows immediately from (21) by replacing $\Upsilon_i(t)$ given by (23).

Corollary 8.1. For each real (nxn) matrix A with $\bar{A}$ Hurwitzian there exist $\alpha>0, \beta>0$ such that

$$|e^{At}| \leq e^{\bar{A}t} \leq \alpha\underline{\alpha}' e^{-\beta t} \text{ for each } t\geq 0.$$

$\underline{\alpha}'$ is the row vector $(\alpha_1^{-1},\ldots,\alpha_n^{-1})$.

Proof. The left-hand inequality is already proved (see (18)). If $\bar{A}$ is Hurwitzian, then, according to Theorem 7 and Proposition 2, system (12) is componentwise exponential asymptotically stable. In view of Definition 4 and taking $t_o=0$ one may write $|e^{\bar{A}t}y_o|\leq \alpha e^{-\beta t}$ for each $t\geq 0$. Replacing now y_o successively by $(\alpha_1,0,\ldots,0),\ldots,(0,\ldots,0,\alpha_n)$ one deduces easily the right-hand inequality.

Remark 5. For fixed a_{ij}, $i,j=1,\ldots,n$, the maximum value of β depends on $\alpha_i>0$, $i=1,\ldots,n$. As a matter of fact one can define the function

$$\beta_{max}(\alpha_1,\ldots,\alpha_n)=\min_i(-a_{ii}-\alpha_i^{-1}\sum_j |a_{ij}|\alpha_j). \quad (26)$$

An equivalent statement for Theorem 8 is the following.

Theorem 9. A necessary and sufficient condition such that (1) be componentwise exponential asymptotically stable is to exist $\alpha>0$ such that

$$\bar{A}\alpha<0. \quad (27)$$

Proof. Sufficiency. If (27) with $\alpha>0$ holds, then there exists $\beta_{max}(\alpha_1,\ldots,\alpha_n)>0$ such that (25) is verified. Necessity is obvious.

Remark 6. If we try to use Theorem 9 we have to prove an existence result for the inequation (27), or equivalently for

$$\hat{A}\alpha < 0, \quad (28)$$

where $\hat{A}=[\bar{A} \mid -I_n]'$, I_n is the unit (nxn) matrix and $[\cdot]'$ signifies the transposition.

1° An existence condition for the problem (27) may be easily derived from Theorem 7 and Proposition 2.

2° For the problem (27) we can also use the notion of M-matrix (Ostrowski, 1955). An M-matrix is defined to be a real (nxn) matrix C such that $c_{ij}\leq 0$, $i\neq j$, possessing one of the following equivalent properties : a) There exists $v>0$ such that $Cv>>0$; b) C is nonsingular and all elements of C^{-1} are nonnegative; c) All principal minors of C are positive. The utility of this definition lies in the following : A necessary and sufficient condition for the existence of a solution $\alpha>0$ for (27) is that $-\bar{A}$ be an M-matrix. Note that the properties b) and c) allow to construct a solution $\alpha>0$ for (27) (directly with b) and via the Gaussian elimination process with c)). In terms of property c) one can estimate the maximum positive eigenvalue $1/\beta_{max}(\alpha_1,\ldots,\alpha_n)$ of $(-\bar{A})^{-1}\geq 0$ by using methods derived from the theorem of Perron - Frobenius.

3° An existence result concerning problem (28), which allows also to construct a solution α, was proved by Dines (1918-1919) : A necessary and sufficient condition for the existence of a solution α for (28) is that the I-rank of $\hat{A}$ be greater than zero. For the definition of the I-rank of a (mxn) real matrix see Dines (1918-1919) or Tschernikow (1971).

In terms of Remark 6 (2° and 3°) and for computational purposes one may prove.

Theorem 10. A necessary and sufficient condition such that (1) be componentwise exponential asymptotically stable is that

$$(-1)^k \bar{A}_k > 0, \quad k=1,\ldots,n.$$

$\bar{A}_1,\ldots,\bar{A}_n$ are the principal minors of $\bar{A}$.

Theorem 11. A necessary and sufficient condition such that (1) be componentwise exponential asymptotically stable is that the I-rank of $\hat{A}$ be greater than zero.

Remark 7. Let us consider

$$A_a=\mathrm{diag}(a_1^{-1},\ldots,a_n^{-1})A\,\mathrm{diag}(a_1,\ldots,a_n),$$

where $a_i\neq 0$, $i=1,\ldots,n$, are the components of the vector a. Notice that A_a is simirespect the solution consists in the assignment of the α - Gershgorin's discs in the half complex plane Re $s < 0$.

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lar to A, both having the same spectrum $\sigma(A)$. The a - Gershgorin's discs associated to A, i.e. the Gershgorin's discs associated to A_a (Bellman, 1960) are subsets of the complex plane C described by

$$G_{Ai}(a) = \{s \in C; |s-a_{ii}| \leq a_i^{-1} \sum_j |a_{ij}| a_j\}, \quad i=1,..,n, \tag{30}$$

which have the remarkable property $\sigma(A) \subset G_A(a)$. $G_A(a)$ represents the union set of $G_{Ai}(a)$, $i=1,..,n$.

Theorem 12. A necessary and sufficient condition such that (1) be componentwise exponential asymptotically stable is that

$$G_A(a) \subset \{s \in C; \text{Re } s < 0\} \tag{31}$$

for at least one $a > 0$.

Proof. For $a = \alpha$ (α from Definition 4), in keeping with (30), condition (31) is equivalent to $\beta_{max}(\alpha_1,..,\alpha_n) > 0$ (see (26)), respectively to (25).

Remark 8. By analogy with the well-known stabilization problem one can state the componentwise stabilization problem of the system $\dot{x}=Ax+Bu$, where $u \in R^m$ and B is a (nxm) real constant matrix. According to Theorem 12 the solution of this problem consists in the assignment of the α- Gershgorin's discs in the half complex plane Re $s<0$ via an adequate state feedback.

DEPENDENCE ON VECTOR BASIS

The componentwise asymptotic stability implies the asymptotic stability. Consequently each of the Theorems 4, 5, 7 - 12 is (mutatis mutandis) a criterion for asymptotic stability. In this respect we remark that the criteria which correspond to Theorems 10 and 11 may be useful in some applications.

The asymptotic stability does not imply the componentwise asymptotic stability, because the latter depends on the particular choice of the vector basis for (1) in R^n. In other words, there exist vector basis in R^n for which the flow invariance of (3) cannot be realised for any $I(t)$, under (19). A natural question is that of the existence of some vector basis in R^n for which an asymptotically stable system is also componentwise asymptotically stable with respect to $\Upsilon(t)$.
A partial answer to this question is the following.

Theorem 13. There exists at least one nonsingular transformation $\tilde{x}=Px$ for (1) such that the system $\dot{\tilde{x}}=\tilde{A}\tilde{x}$, $t \geq 0$, be componentwise asymptotically stable with respect to $\Upsilon(t)$ if

$$\sigma(A) \subset \{s \in C; \text{Re } s<0, |\text{Im } s| < -\text{Re } s\}. \tag{32}$$

Proof. Consider $P=V^{-1}$, where V is the modal matrix of A over R. Since $\tilde{A}=V^{-1}AV$ is the (block) diagonal or the (block) Jordan canonical form of A over R, according to (32), it follows that $\bar{\tilde{A}}$ is Hurwitzian.

EXAMPLE

Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} x, \quad t \geq 0, \quad x(0)=x_o, \tag{33}$$

and find a function $\Upsilon(t)$ such that (33) be componentwise asymptotically stable.

Solution. Since $\bar{A}$ is Hurwitzian one may adopt $\Upsilon(t)=e^{\bar{A}t}\Upsilon(0)$, which satisfies Theorem 4. One may easily see that

$$\Upsilon(t)=\frac{1}{6}\begin{bmatrix} ce^{-bt}+de^{-at} & fe^{-bt}-fe^{-at} \\ ge^{-bt}-ge^{-at} & de^{-bt}+ce^{-at} \end{bmatrix}\Upsilon(0), t \geq 0,$$

where $a=2+\sqrt{3}$, $b=2-\sqrt{3}$, $c=3+\sqrt{3}$, $d=3-\sqrt{3}$, $f=2\sqrt{3}$, $g=\sqrt{3}$. For $\Upsilon_1(0)=(a-1)\rho$ and $\Upsilon_2(0)=\rho$, with arbitrary $\rho>0$, we obtain

$$\Upsilon(t) = \rho[a-1 \quad 1]' \, e^{-bt}, \quad t \geq 0,$$

which proves that system (33) is also componentwise exponential asymptotically stable.

CONCLUSIONS

The notion of flow invariance proves to be an efficient tool for a more subtle characterization of temporal behaviour of the linear constant dynamical systems. In fact, when the flow invariant set is a time-dependent state interval $I(t)$, a componentwise characterization of the free response is possible. Such an evaluation may be useful especially when the state components are of different importance for the normal evolution of the systems (for instance in Electrical Engineering (Voicu, 1984) or in Biology (Pavel, 1983)). The simply necessary and sufficient condition (9) allows to determine $\Upsilon(t)$ for which $I(t)$ is globally flow invariant for a given linear constant dynamical system or to determine the matrix A such that a given $I(t)$ be globally flow invariant.

The componentwise asymptotic stability of (1), which is a globally flow invariance of $I(t)$ for (1), with $I(t) \to 0$ as $t \to \infty$, represents a row property of the evolution matrix A consisting in a certain first-diagonal dominance (see (21)), and holds if and only if $\bar{A}$ is Hurwitzian. This special type of asymptotic stability depends on the vector basis and it implies the asymptotic stability in the sense of Liapunov.

The results of the paper may be also used for the linear state feedback synthesis (Voicu, 1981; Voicu, 1983) in the componentwise stabilization problem. In this