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ON THE APPLICATION OF THE FLOW-INVARIANCE METHOD IN CONTROL THEORY AND DESIGN

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Abstract. The purpose of this paper is to demonstrate the powerfulness of the flow-invariance method for a more subtle characterization of the nonlinear control systems. The crucial premise is that the state flow-invariant set is a rectangular and time-dependent box which specifically allows a componentwise characterization of the system evolution. The main results concern the control and state componentwise constrained evolution and the componentwise stability. They are expressed by necessary and sufficient inequality conditions. As such the constrained evolution and the componentwise stability may be designed in a natural way as robust properties. For nonlinear matrix systems one introduces the componentwise absolute stability by analogy with the classical one. Its characterization by an adequate linearly majorant system and the subsequent componentwise evaluation of the solution by an exponentially decaying and positive vector may be significantly interesting in some application fields such as engineering and economics, ecology, arms races, pharmacokinetics, transistor circuits etc.

Keywords. Nonlinear control systems ; componentwise constrained evolution ; robustness ; componentwise asymptotic stability ; componentwise absolute stability.

INTRODUCTION

Consider the nonlinear continuous-time dynamical system

$$\dot{x} = f(t,x,u), \quad t \in R_+, \quad x \in R^n, \quad u \in R^m, \quad (1)$$

where x is the state, u is the control and f is a vectorial function.
In compliance with the usually specified attributes of a dynamical system we assume that $u \in \overline{C}^o$, where $\overline{C}^o$ is the set of the piecewise continuous functions in

$$T \triangleq [t_o, t_1) \subseteq R_+, \quad (2)$$

that is, continuous except a number of points in T, all being left discontinuity points only. It is known that for such controls one has to approach the state controllability and especially the optimal control problem for the system (1). As far as it concerns these problems, according to some practical reasons, one ascertains that u has to belong only to a certain set of admissible controls

$$\mathcal{U} \triangleq \{u \in \overline{C}^o \ ; \ u(t) \in U, \quad t \in T\}, \quad (3)$$

where U is a compact subset of R^m. Similarly, by reviewing the aspects regarding the state, it also results that x(t) has to remain for all $t \in T$ in a certain compact subset $X \subset R^n$ only.
Under these circumstances the statement of the characterization problem concerning the evolution of system (1) on $T \times \mathcal{U} \times X$ is consistent and of practical interest of course.

In this paper we will show that such a characterization may be accomplished by using the flow-invariance method for a special form of X, which seems to be adequate for many applications. The origin of the flow-invariance method goes back to the results of Nagumo (1942), Hukuhara (1954), Brezis (1970), Crandall (1972) and Martin jr. (1973). An essential exposition in a coherent way of the main recent research in this field is due to Pavel (1984). The significant results of this paper lean upon the Lemma 4.2 (Pavel, 1984), due to Pavel and Voicu (1986), which assumes that X is a rectangular and time-dependent box in R^n.

CONTROL AND STATE CONSTRAINED EVOLUTION

Nonlinear Control Systems

We begin with some notations and definitions. Let $v \triangleq (v_i)$ and $w \triangleq (w_i)$ be two vectors in R^k. In all that follows we denote by $|v|$ the vector with the components $|v_i|$ and by $v \leq w$ ($v < w$) or by $v \geq w$ ($v > w$) the inequalities $v_i \leq w_i$ ($v_i < w_i$) or $v_i \geq w_i$ ($v_i > w_i$) respectively.

Let $V \subset R^n$ be a compact subset and $\varphi: V \to R^n$ be a continuous function with the components φ_i and let $z \triangleq (z_1,\ldots,z_n)$ be a certain point in V. We denote by $\mathcal{C}_v^z$ the operator which "catches" $\varphi(v)$ at $z \in V$ in a diagonal manner, i.e. $\mathcal{C}_v^z\{\varphi(v)\} \triangleq [\varphi_1(z_1,v_2,\ldots,v_n),\ldots,\varphi_i(v_1,\ldots,z_i,\ldots,$

$v_n), \ldots, \varphi_n(v_1, \ldots, v_{n-1}, z_n)]'$ where $[\cdot]'$ signifies the transposition. We also denote by $\operatorname{ext}_V \mathcal{C}_v^z\{\varphi(v)\}$, where ext may be min or max, the vector with the components $\operatorname{ext}_V \varphi_i(v_1, \ldots, z_i, \ldots, v_n)$.

Let $\underline{a} : T \to R^n$ and $\overline{a} : T \to R^n$ be two differentiable functions such that $\underline{a}(t) \leq \overline{a}(t)$, $t \in T$, and let $\underline{b} : T \to R^m$ and $\overline{b} : T \to R^m$ be two continuous functions such that $\underline{b}(t) \leq \overline{b}(t)$, $t \in T$. It is known that usually both x and u are subject to certain prescribed constraints due to physical and/or constructive reasons. We define the state and the control constraints associated to the dynamical system (1) in the form of the following two rectangular and time-dependent boxes :

$$X(t) \triangleq \{v \in R^n \; ; \; \underline{a}(t) \leq v \leq \overline{a}(t)\}, \; t \in T, (4)$$

$$U(t) \triangleq \{w \in R^m \; ; \; \underline{b}(t) \leq w \leq \overline{b}(t)\}, \; t \in T. (5)$$

Definition 1. The dynamical system (1) is called $T\mathcal{U}X$ - constrained evolutionally if for each $x(t_o) \triangleq x_o \in X(t_o)$ and for each $u \in \mathcal{U}$ the response of (1) satisfies

$$x(t) \in X(t) \text{ for each } t \in T. \qquad (6)$$

Remark 1. According to Ursescu (1982) the $T\mathcal{U}X$ - constrained evolution of (1) for arbitrary compact sets X(t) and U(t), in the condition of unicity of the Cauchy solution (i.e. f is continuous in t, x and u and locally Lipschitzian in x), is equivalent to the flow-invariance of X(t) for each $u \in \mathcal{U}$ on T, respectively to

$$\liminf_{h \searrow 0} h^{-1} d(v+hf(t,v,u(t)); X(t+h)) = 0 \; (7)$$

for each $(t,u,v) \in T \times \mathcal{U} \times X$. $d(v;V)$ signifies the distance from $v \in R^n$ to the set $V \subset R^n$.

Remark 2. For the state and control constraints the rectangular boxes (4) and (5) may be substituted by the curved boxes $X(t) \triangleq \{v \in R^n \; ; \; \underline{a}(t) \leq p(v) \leq \overline{a}(t)\}$ and $U(t) \triangleq \{w \in R^m \; ; \; \underline{b}(t) \leq q(w) \leq \overline{b}(t)\}$ respectively, where $p : R^n \to R^n$ and $q : R^m \to R^m$ must be invertible and derivable and respectively invertible and continuous. Under such hypotheses the $T\mathcal{U}X$ - constrained evolution, by using the transformations $\tilde{x} = p(x)$ and $\tilde{u} = q(u)$, is equivalent to the $T\tilde{\mathcal{U}}\tilde{X}$ - constrained evolution, where $\tilde{X}(t)$ and $\tilde{U}(t)$ are both rectangular boxes.

Remark 3. The time-dependent boxes (4) and (5) generalize the well known componentwise constraint $|u(t)| \leq u_o$ = constant, which is the standard start point for the Pontriaghin's principle. The special form of X(t) allows, as we show below, an explicit analytical conversion of the tangential condition (7), which is very powerful of course, but intricate enough in handling for X(t) of arbitrary form. In contrast, by this conversion the specific form of U(t), for instance the form (5), does not play any role. Consequently the following result concerning the $T\mathcal{U}X$ - constrained evolution refers to a certain unspecified compact U(t), except the situations when (5) is deliberately considered.

Theorem 1. The dynamical system (1), with f continuous and locally Lipschitzian, is $T\mathcal{U}X$ - constrained evolutionally if and only if

$$\min_{T \times \mathcal{U} \times X} \left[\mathcal{C}_v^{\underline{a}(t)}\{f(t,v,u(t))\} - \dot{\underline{a}}(t)\right] \geq 0, (8)$$

$$\max_{T \times \mathcal{U} \times X} \left[\mathcal{C}_v^{\overline{a}(t)}\{f(t,v,u(t))\} - \dot{\overline{a}}(t)\right] \leq 0. (9)$$

Proof. On the basis of Remark 1 and of Lemma 4.2 (Pavel, 1984) it follows that the $T\mathcal{U}X$ - constrained evolution of system (1), in view of differentiability of $\underline{a}(t)$ and $\overline{a}(t)$, is equivalent to

$$\underline{a}(t) + h\dot{\underline{a}}(t) + h\underline{\alpha}(h) \leq$$
$$\leq v + hf(t,v,u(t)) + h\alpha(h) \leq \qquad (10)$$
$$\leq \overline{a}(t) + h\dot{\overline{a}}(t) + h\overline{\alpha}(h),$$

for each $(t,u,v) \in T \times \mathcal{U} \times X$, for $h > 0$ small enough and for certain functions $\underline{\alpha} : T \to R^n$, $\alpha : T \to R^n$ and $\overline{\alpha} : T \to R^n$, with $\underline{\alpha}(h) \to 0$, $\alpha(h) \to 0$ and $\overline{\alpha}(h) \to 0$ as $h \searrow 0$.

Substituting v successively by $(\underline{a}_1(t), v_2, \ldots, v_n), \ldots, (v_1, \ldots, \underline{a}_i(t), \ldots, v_n), \ldots, (v_1, \ldots, v_{n-1}, \underline{a}_n(t))$, where $\underline{a}_i(t)$ are the components of $\underline{a}(t)$, and by $(\overline{a}_1(t), v_2, \ldots, v_n), \ldots, (v_1, \ldots, \overline{a}_i(t), \ldots, v_n), \ldots, (v_1, \ldots, v_{n-1}, \overline{a}_n(t))$ respectively, where $\overline{a}_i(t)$ are the components of $\overline{a}(t)$, and simplifying by $h > 0$ it follows that (10), under $h \searrow 0$, is equivalent to

$$\mathcal{C}_v^{\underline{a}(t)}\{f(t,v,u(t))\} \geq \dot{\underline{a}}(t), \qquad (10a)$$

$$\mathcal{C}_v^{\overline{a}(t)}\{f(t,v,u(t))\} \leq \dot{\overline{a}}(t) \qquad (10b)$$

both for each $(t,u,v) \in T \times \mathcal{U} \times X$. The necessity part of this equivalence is obvious. To prove the sufficiency part of this equivalence assume by contradiction that (10a) is not true, that is $f_i(t, v_1, \ldots, v_{i-1}, \underline{a}_i(t), v_{i+1}, \ldots, v_n, u(t)) < \dot{\underline{a}}_i(t)$, where f_i is a certain component of f. Then it is easy to prove that for $h > 0$, small enough, we obtain

$$h^{-1} d([v_1 \ldots v_{i-1} \; \underline{a}_i(t) \; v_{i+1} \ldots v_n]' + h[\underline{f}_1 \ldots \underline{f}_i \ldots \underline{f}_n]' \; ; \; X(t+h)) = |\underline{f}_i - \dot{\underline{a}}_i(t)| \neq 0 \text{ for}$$

$v_j \in (\underline{a}_j(t), \overline{a}_j(t))$, $j=1, \ldots, n$, $j \neq i$, with $\underline{f}_k \triangleq f_k(t, v_1, \ldots, v_{i-1}, \underline{a}_i(t), v_{i+1}, \ldots, v_n, u(t))$, $k=1, \ldots, n$, which contradicts (7). Similarly, one can prove that if (10b) is not true, then (7) is also not true.

Now it is a simple matter to see that (10a) and (10b) are equivalent to (8) and (9) respectively.

Remark 4. The conditions (8) and (9) represent nonconventionally differential inequalities and they may be used in control design, i.e. for the determination of f for prespecified T, U(t) and X(t). If this problem can be solved, then, taking into account the inequality form of the conditions (8) and (9), we may suppose that there exists a class of solutions f. Consequently the designer may adopt a suitable solution f, in accordance also with other supplementary conditions, but such that for its imperfect practical realization and/or in the presence of some uncertainties (both holding between certain limits) the conditions (8) and (9) still remain valid. As such the $T\mathcal{U}X$ - constrained evolution of the dynamical system (1) may be designed from the start point and in a natural way as a robust property. Thus a relevant application field seems to be the design of the spatial manipulating systems (Voicu, 1985).

<u>Linear Constant Control Systems</u>
In this case the conditions (8) and (9) become transparent enough and relatively easy in handling so that they may be more significant for applications.
Let us consider the linear constant dynamical system

$$\dot{x} = Ax + Bu, \quad t \in R_+, \quad x \in R^n, \quad u \in R^m, \quad (11)$$

where A and B are real constant matrices of adequate dimensions.
Clearly we may assume for U(t) (see (5)) that

$$-\underline{b}(t) = \overline{b}(t) \triangleq b > 0, \qquad t \in R_+, \quad (12)$$

and $T = R_+$. According to (12) the system (11) possesses a symmetry point corresponding to $u = 0$ and $x = 0$. Thus we may further assume for X(t) (see (4)) that

$$-\underline{a}(t) = \overline{a}(t) \triangleq a > 0, \qquad t \in R_+. \quad (13)$$

<u>Remark 5</u>. Under these hypotheses the state and the control constraints are

$$|x(t)| \leqslant a, \qquad |u(t)| \leqslant b, \qquad t \in R_+,$$

and instead of the $R_+\mathcal{U}X$ - constrained evolution of (11) we can simply and specifically talk about <u>the componentwise constrained evolution</u> (CCE) of (11). Actually such a peculiar evolution may be also considered for the dynamical system (1).

For the sake of simplicity of writing we introduce some auxiliary notations. Let $M \triangleq (m_{ij})$ be a real (kxr) matrix. We denote by $|M|$ the matrix with the elements $|m_{ij}|$ and by $\overline{M}$, for k=r, the matrix with the elements m_{ii} and $|m_{ij}|$, $i \neq j$.

<u>Theorem 2</u>. The dynamical system (11) is CCE if and only if

$$\overline{A}a + |B|b \leqslant 0. \quad (14)$$

<u>Proof</u>. The extrema involved in (8) and (9), with (4), (13) and (5), (12), can be calculated as follows

$$-\underset{R_+ \times \mathcal{U} x X}{\min} \mathcal{C}_v^{-a}\{Av + Bu(t)\} = \underset{R_+ x \mathcal{U} x X}{\max} \mathcal{C}_v^{a}\{Av + Bu(t)\} = \overline{A}a + |B|b.$$

Clearly, condition (14) depends on the particular choice of the vector bases for x and u. For instance, if A, B are in the controllability canonical form, condition (14) cannot be satisfied. A natural question is that of the existence of vector bases in R^n and R^m respectively for which system (11) is CCE. In order to clarify this, let us consider the transformation $\tilde{x} = V_R^{-1}x$, where V_R is the modal matrix of A over R. Such being the case $V_R^{-1}AV_R$ is the (block) diagonal or the (block) Jordan canonical form of A over R. Obviously the inequality

$$\overline{V_R^{-1}AV_R}a + |V_R^{-1}B|b \leqslant 0,$$

which corresponds to (14) but for the new state vector $\tilde{x}$, can be met if the real parts of the eigenvalues of A are negative and sufficiently small.

COMPONENTWISE ASYMPTOTIC STABILITY

The notion of $T\mathcal{U}X$ - constrained evolution may produce, under certain supplementary assumptions regarding X(t) and U(t), some special type of stability results. For example the CCE (see <u>Remark 5</u>) can be considered as a necessary and sufficient condition for componentwise bounded input ~ bounded state (CBIBS) stability. Similar results may be obtained for the internal stability ($u(t) = 0$, $t \in R_+$). In the case of linear constant dynamical systems such a subject was covered, via flow-invariance method, in other papers (Voicu, 1984a, b). Next we shall firstly investigate the nonlinear case by the same method.

<u>Componentwise Asymptotic Stability</u>
Let us consider that the dynamical system (1) is free ($u(t) = 0$, $t \in R_+$), that is

$$\dot{x} = f(t,x), \quad t \in R_+, \quad x \in R^n, \quad (15)$$

with

$$f(t,0) = 0, \quad t \in R_+.$$

Since $x = 0$ is an equilibrium point of (15) we may consider the rectangular time-dependent box

$$X_o(t) \triangleq \{v \in R^n \; ; \; |v| \leqslant \gamma(t)\}, \quad t \in R_+,$$

where $\gamma: R_+ \to R^n$ is differentiable, with $\gamma(t) > 0$, $t \in R_+$.
As far as it concerns the stability of $x = 0$ we have furthermore to require that

$$\lim_{t \to \infty} \gamma(t) = 0. \quad (16)$$

<u>Definition 2</u>. The point $x = 0$ is called componentwise asymptotically stable with respect to $\gamma(t)$ (CWASγ) if for each $t_o \geqslant 0$ and for each x_o, with $|x_o| \leqslant \gamma(t_o)$, the response of (15) satisfies

$$|x(t)| \leqslant \gamma(t), \quad t \geqslant t_o. \quad (17)$$

The corresponding characterization of system (15) can be easily derived from <u>Theorem 1</u>.

<u>Theorem 3</u>. The point $x = 0$ is CWASγ if and only if

$$\underset{t \geqslant 0, |v| \leqslant \gamma}{\max} \left[\mathcal{C}_v^{\pm\gamma}\{\pm f(t,v)\} - \dot{\gamma}(t)\right] \leqslant 0. \quad (18)$$

Note that the signs preceding γ and f in (18) must be in concordance.

<u>Remark 6</u>. It is obvious that condition (17), under (16), can be satisfied only if the equilibrium point $x = 0$ is asymptoti-

cally stable in Liapunov sense. Consequently inequality (18) represents a sufficient condition for the asymptotic stability of the point x = 0. Moreover, the set $X_{oAS} \triangleq \{ v\in R^n ;\ |v| \leq \max_{R_+} \gamma(t)\}$, is one of its asymptotic stability region.

In this context we can state the question of the global CWASγ. Clearly one can identify many premises for its consistent definition. It seems to us that the following is natural and simple.

Definition 3. Assume that $\gamma(t)$ in Definition 2 is replaced with $\rho\gamma(t)$, $\rho \geq 1$. The point x = 0 is called globally CWASγ (or simply the system (15) is CWASγ) if x= 0 is CWASγ for all $\rho \geq 1$.

The corresponding characterization of system (15) may be derived with Theorem 3 (mutatis mutandis).

Theorem 4. The dynamical system (15) is CWASγ if and only if

$$\max_{\substack{t\geq 0\\ |v|\leq\gamma,\ \rho\geq 1}} \left[\frac{1}{\rho}\mathcal{C}^{\pm\gamma}_{v}\{\pm f(t,\rho v)\} - \dot{\gamma}(t)\right] \leq 0. \quad (19)$$

Remark 7. Following Remark 6 we may assert that (19) is a sufficient condition for the asymptotic stability in the large of x = 0.

Assuming that the dynamical system (15) is linear and constant, i.e.

$$\dot{x} = Ax, \quad t\in R_+, \quad x\in R^n, \quad (20)$$

the specialized form of Theorem 4 is the following.

Theorem 5. The dynamical system (20) is CWASγ if and only if

$$\max_{R_+}\left[\bar{A}\gamma(t) - \dot{\gamma}(t)\right] \leq 0. \quad (21)$$

Proof. Observe that the maximum from (19) for $|v| \leq \gamma(t)$ is to be determined by the proof of Theorem 2.

Componentwise Exponential Asymptotic Stability

As we have already proved (Voicu, 1984a; Proposition 2) CWASγ of system (20) is equivalent to CWEAS (E = exponential) of the same system. The corresponding definitions of the second stability type, regarding (15), are the following.

Definition 4. The point x = 0 is called CWEAS if there exist $\alpha > 0$ (vector with the components α_i) and $\beta > 0$ (scalar) such that for each $t_o\in R_+$ and for each x_o, with $|x_o| \leq \alpha$, the response of (15) satisfies

$$|x(t)| \leq \alpha e^{-\beta(t-t_o)}, \quad t\geq t_o. \quad (22)$$

Definition 5. Assume that α in Definition 4 is replaced with $\rho\alpha$, $\rho \geq 1$. The equilibrium point x = 0 is called globally CWEAS (or simply the system (15) is CWEAS) if x = 0 is CWEAS for all $\rho \geq 1$.

Obviously the characterizations via Theorems 3 and 4 may be derived by specializing $t_o = 0$, $\gamma(t) = \alpha e^{-\beta t}$ and by replacing v with $ve^{-\beta t}$.

Theorem 6. The point x = 0 is CWEAS if and only if

$$\max_{\substack{t\geq 0\\ |v|\leq\alpha}} \left[e^{\beta t}\mathcal{C}^{\pm\alpha}_{v}\{\pm f(t,ve^{-\beta t})\}\right] \leq -\beta\alpha. \quad (23)$$

Theorem 7. The dynamical system (15) is CWEAS if and only if

$$\max_{\substack{t\geq 0\\ |v|\leq\alpha,\ \rho\geq 1}} \left[\frac{1}{\rho}e^{\beta t}\mathcal{C}^{\pm\alpha}_{v}\{\pm f(t,\rho ve^{-\beta t})\}\right] \leq -\beta\alpha. \quad (24)$$

For the further development let us recall the characterization of (20) regarding CWEAS. Firstly some notations and definitions : $A \triangleq (a_{ij})$; $A_\alpha \triangleq \mathrm{diag}(1/\alpha_1,\dots,1/\alpha_n)\, A\, \mathrm{diag}(\alpha_1,\dots,\alpha_n)$; $G_i(A_\alpha) \triangleq \{s\in C ;\ |s - a_{ii}| \leq \frac{1}{\alpha_i}\sum_{j=1,n;\ j\neq i}|a_{ij}|\alpha_j\}$, i=1,...,n, which are the α- Gershgorin's disks associated to A (Bellman, 1960 ; p. 107) ; $\bar{A}_k$, k=1,...,n, are the leading principal minors of $\bar{A}$; M > 0 simbolizes an elementwise inequality (i.e. $m_{ij} > 0$).

Theorem 8. For the dynamical system (20) the following statements are equivalent : 1° (20) is CWEAS ; 2° $\bar{A}\alpha \leq -\beta\alpha$; 3° $0 < \beta \leq \min_i(-a_{ii} - \frac{1}{\alpha_i}\sum_{j=1,n;\ j\neq i}|a_{ij}|\alpha_j)$; 4° $\bar{A}\alpha < 0$; 5° $-\bar{A}$ is an M-matrix (Bellman, 1960 ; p.295) ; 6° $\bar{A}$ is Hurwitzian ; 7° $\bigcup_{i=1}^{n} G_i(A_\alpha) \subset \{s\in C ;\ \mathrm{Re}\, s < 0\}$; 8° $(-1)^k \bar{A}_k > 0$, k=1,...,n ; 9° det A ≠ 0, $(-\bar{A})^{-1} \geq 0$.

To prove the equivalence between 1° and 2° one applies Theorem 7. For the other equivalences see for instance Voicu (1984a, b).

As it has already been underlined in these two papers, CWEAS of (20) is a special type of asymptotic stability, depending on the vector basis in R^n. This represents a row property of A, which holds if and only if $\bar{A}$ satisfies one of the statements 2° - 9° from Theorem 8. CWEAS of (20) corresponds to a certain dominance of the first diagonal elements of A in the row direction, necessarily implying that these elements are negative (see 3° from Theorem 8).

Componentwise Absolute Stability

The inequality form of condition (24) and Theorem 8 suggests a peculiar approach of a class of nonlinear dynamical systems (15) in the situation when they may be expressed by the equation

$$\dot{x} = F(t,x)x, \quad t\in R_+, \quad x\in R^n. \quad (25)$$

F(t,x) belongs to a class of (nxn) matrices which are continuous and adequately bounded. For our purpose this boundness must be understood in the following sense: for a given real constant (nxn) matrix A there exist $\alpha > 0$, $\beta > 0$ such that the following elementwise inequality holds

$$\mathcal{C}^{\pm\alpha}_{v}\{F(t,\rho ve^{-\beta t})\} \leq \bar{A},\ t\geq 0,\ |v|\leq\alpha,\ \rho\geq 1, \quad (26)$$

where $\mathcal{C}^{\pm\alpha}_{v}$ is to apply to each column of F.

Remark 8. Clearly there exists a nonempty class $\mathcal{F}_{\bar{A}}$ of continuous matrices F(t,x) which may satisfy (26). Under these circumstances the linear constant dynamical system (20) may be called the linear elementwise $\mathcal{C}$- majorant of all the dynamical systems (25) with $F \in \mathcal{F}_{\bar{A}}$.

Definition 6. The dynamical system (25) is called componentwise absolutely stable if it is CWEAS for all $F \in \mathcal{F}_{\bar{A}}$.

The corresponding characterization of (25) is the following.

Theorem 9. The dynamical system (25) is componentwise absolutely stable if and only if its linear elementwise $\mathcal{C}$- majorant (20) is CWEAS.

Proof. Sufficiency. First note that for $t \geqslant 0$, $|v| \leqslant \alpha$, $\rho \geqslant 1$ and $F \in \mathcal{F}_{\bar{A}}$ we have

$$\mathcal{C}^{\pm\alpha}_{v}\{\pm F(t,\rho v e^{-\beta t})v\} \leqslant \mathcal{C}^{\pm\alpha}_{v}\{F(t,\rho v e^{-\beta t})\alpha\} = \mathcal{C}^{\pm\alpha}_{v}\{\bar{F}(t,\rho v e^{-\beta t})\}\alpha \leqslant \bar{A}\alpha . \quad (27)$$

If (20) is CWEAS, according to (27) and to 2° from Theorem 8, one may write

$$\mathcal{C}^{\pm\alpha}_{v}\{\pm F(t,\rho v e^{-\beta t})v\} \leqslant \bar{A}\alpha \leqslant -\beta\alpha$$

for $t \geqslant 0$, $|v| \leqslant \alpha$, $\rho \geqslant 1$ and $F \in \mathcal{F}_{\bar{A}}$. In view of Theorem 7 (applied to system (25)) and Definition 6 it follows that system (25) is componentwise absolutely stable.
Necessity is obvious by adopting $F(t,x) = \bar{A}$.

Remark 9. For a class of nonlinear matrix systems of the form (25) in engineering and economics, ecology, arms races, pharmacokinetics, transistor circuits etc a study of "the asymptotic connective stability" via the Liapunov direct method is due to Šiljak (1975a, b). By defining the class $\mathcal{M}_{\bar{M}}$ of continuous matrices for which $\bar{F}(t,x) \leqslant \bar{M}$ for $t \in R_+$, $x \in R^n$, where M is a given real constant (nxn) matrix, one characterizes (25) in terms of "the absolute exponential and connective stability (AECS)" and gives the time-domain evaluation

$$\|x(t)\| \leqslant k e^{-\varepsilon(t-t_o)}, \quad t \geqslant t_o, \quad (28)$$

where $\|\cdot\|$ is the Euclidean norm, $k > 0$ and $\varepsilon > 0$. In this case system (25) is AECS if and only if $\bar{M}$ is Hurwitzian (for other equivalent conditions, mutatis mutandis, see Theorem 8).
It must be pointed out here that if (25) is CWEAS then it is also exponentially and connectively stable because the more detailed time-domain evaluation (22) implies (28), i.e. the first is stronger than the second. In spite of this for a given $\bar{A}$ the class $\mathcal{F}_{\bar{A}}$ seems to be larger than $\mathcal{M}_{\bar{A}}$.
To justify this last assertion let us examine an adequate example.

Example. Consider the dynamical system (25) with n=1 and $\bar{A} = -1$. If we adopt $F(t,x) = -e^{2t}|x|$, then there exist $\alpha = 1$ and $\beta = 1$ such that $F \in \mathcal{F}_{-1}$ because $-\rho e^{2t}e^{-t} \leqslant -1$, $t \in R_+$, $\rho \geqslant 1$. Consequently $|x(t)| \leqslant \rho e^{-(t-t_o)}$, $t \geqslant t_o \geqslant 0$, for each $|x_o| \leqslant \rho$, $\rho \geqslant 1$, i.e. the system is CWEAS or equivalently (owing to n=1) exponentially and connectively stable. It is a simple matter to verify this by using

$$x(t) = 2x_o/(2+|x_o|(e^{2t}-e^{2t_o})), \quad t \geqslant t_o, x_o \in R,$$

which is the Cauchy solution of the system

$$\dot{x} = -e^{2t}|x|x.$$

At the same time $F \notin \mathcal{M}_{-1}$ because $-e^{2t}|x| \leqslant 0$, $t \in R_+$, $x \in R$, i.e. $F \in \mathcal{M}_0$; obviously $\mathcal{M}_0$ cannot be a class for which the system be AECS.

CBIBS Stability
We shall return to the linear constant dynamical system (11). Clearly it depends on A and B whether condition (14) can or cannot be satisfied for prescribed a and b. If we try to determine a and b for given A and B we have to prove an existence result for the inequation (14). A partial answer in this respect is the following.

Theorem 10. A necessary and sufficient condition such that for each B and for each b to exist an a such that the dynamical system (11) be CCE (or equivalently CBIBS stable) is that (11) be CWEAS.

Proof. Sufficiency. If (11) is CWEAS then, according to 4° and 9° from Theorem 8, for each B and for each b there exists

$$a = \varepsilon_1(-\bar{A})^{-1}|B|b + \varepsilon_2\alpha > 0,$$

where $\varepsilon_1 \geqslant 1$ and $\varepsilon_2 > 0$ are two arbitrary scalars, for which the inequation (14) is satisfied.
Necessity. If (14) holds for each B and for each b then $\bar{A}a < 0$, which suffices for CWEAS of (11).

Componentwise Stabilization
Consider the linear constant dynamical system (11) and let

$$u = -Kx + v \quad (29)$$

be a linear state feedback and control law, where K is a real constant (mxn) matrix and $v \in R^m$.
By replacing (29) into (11) one obtains

$$\dot{x} = Fx + Bv, \quad t \in R_+, \quad x \in R^n, \quad v \in R^m, \quad (30)$$

where $F \triangleq A - BK$.
In view of Theorem 10 the following problem may be of practical interest : "determine K such that (30) be CWEAS for prescribed α and β". Following 3° and 7° from Theorem 8 we can say that this problem consists in the α - Gershgorin's disks assignment in the complex half plane Re $s \leqslant -\beta < 0$. As such one cannot approach it as a problem of explicit pole assignment, but one must solve the algebraic nonlinear system

$$\bar{F}\alpha \leqslant -\beta\alpha \quad (31)$$

$$BK = A - F \quad (32)$$

for given A, B and α, β.

Assume that B is of full rank and $m < n$. This implies that there exists a real constant and nonsingular (nxn) matrix P such that

$$PB = [B_m', \ 0]' ,$$

where B_m is a certain nonsingular (mxm) matrix (Bouillon and Odell, 1971).
By multiplying left with P in (32) and partitioning P as PB, i.e.

$$P \triangleq [P'_m, P'_{n-m}]' ,$$

one obtains from (32) a solution

$$K = B_m^{-1} P_m (A - F) \qquad (33)$$

if and only if the consistency condition

$$P_{n-m}(A - F) = 0 \qquad (34)$$

holds. Thus the algebraic nonlinear system (31), (32) has a solution (33) if and only if the system (31), (34) admits a solution F for at least one P_{n-m}.

Remark 10. If we consider that K is given and B is the unknown, then the nonlinear problem (31), (32) can be easily reduced, in a certain sense, to the linear case. Such a problem is considered in an other paper (Voicu, 1987) and specifically it corresponds to the componentwise state detection, i.e. detection by observing the state with componentwise absolutely and exponentially decaying error.

CONCLUDING REMARKS

The main conclusion of this paper is that one tries to demonstrate the powerfulness of the flow-invariance method for a more subtle characterization of the nonlinear control systems. The crucial premise for this demonstration is that the state flow-invariant set is a rectangular and time-dependent box in R^n which specifically allows an explicit analytical conversion of the tangential condition (7) and by this a componentwise characterization of the system evolution. The obtained results concern the control and state componentwise constrained evolution and the componentwise stability. All these results are necessary and sufficient conditions and they are always expressed by inequalities which may be satisfied for a class of nonlinear control systems. As such the componentwise constrained evolution and the componentwise stability may be designed from the start point and in a natural way as robust properties. Following this idea and by analogy with the classical notion of absolute stability one defines the componenentwise absolute stability for nonlinear matrix systems. Its surprising simple characterization by an adequate linearly majorant system and the subsequent time-domain componentwise evaluation of the Cauchy solution by an exponentially decaying and positive vector may be significantly interesting in some application fields as engineering and economics, ecology, arms races, transistor circuits etc. The results of Šiljak (1975a, b) in this respect do not cover our results because the time-domain evaluation (22) of the Cauchy solution is more detailed than (28) and at the same time the class of nonlinear matrix systems $\mathcal{F}_{\bar{A}}$ seems to be larger than that considered by Šiljak.

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