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Necessary and Sufficient Conditions for Componentwise Stability of Interval Matrix Systems

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***Abstract*—In the asymptotic stability (AS) analysis of interval matrix systems, some results are available that operate only as sufficient conditions, based on a unique test matrix, adequately built from the interval matrix. Our note reveals the complete role of this test matrix for fully characterizing the componentwise asymptotic stability (CWAS) of interval matrix systems. CWAS is a special type of AS which ensures the flow invariance of certain time-dependent sets with respect to the state-space trajectories. Hence, the sufficient conditions for AS get a new and deeper meaning by their reformulation as necessary and sufficient conditions with respect to the stronger property of CWAS.**

***Index Terms*—Asymptotic stability, componentwise asymptotic stability, interval matrix systems, time-dependent invariant sets.**

I. Introduction

In the late 1980s and early 1990s, a large body of work was invested to explore the Hurwitz or Schur *asymptotic stability* (AS) of *interval matrix systems* (IMSs)

$$(x(t))' = Ax(t); x(t_0) = x_0; t, t_0 \in \mathbf{T}, t \geq t_0, A \in A^I \quad (1)$$

with continuous-time ($\mathbf{T} = \mathbf{R}_+$) or discrete-time ($\mathbf{T} = \mathbf{Z}_+$) dynamics and the operator $(\ )'$ acting accordingly, where A^I denotes an interval matrix

$$A^I = \{A \in \mathbf{R}^{n \times n} : A^- \leq A \leq A^+\}. \quad (2)$$

The two matrix inequalities in (2) have the componentwise meaning $a_{ij}^- \leq a_{ij} \leq a_{ij}^+, i, j = 1, \ldots, n$, where $a_{ij}^-, a_{ij}, a_{ij}^+$ represent generic elements of matrices A^-, A, A^+, respectively.

Some of the papers issued during the aforementioned period provide *sufficient conditions* for AS of IMS (1), which are very easy to handle, because their formulation relies on a unique test matrix $\overline{A}$ built from the bounds a_{ij}^-, a_{ij}^+ of A^I. Two remarkable results of this type are given by the following.

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Lemma 1:

i) [1, Cor. 2.2] In the continuous-time case, IMS (1) is asymptotically stable if the test matrix $\overline{A} = (\overline{a}_{ij})_{i,j=1,\ldots,n}$ defined below by (3) is Hurwitz stable

$$\begin{aligned}\overline{a}_{ii} &= \sup_{A \in A^I} \{a_{ii}\} = a_{ii}^+, \qquad i = 1,\ldots,n\\ \overline{a}_{ij} &= \sup_{A \in A^I} \{|a_{ij}|\} = \max\{|a_{ij}^-|, |a_{ij}^+|\}, \qquad i \neq j, i,j = 1,\ldots,n.\end{aligned} \tag{3}$$

ii) [1, Cor. 1.2], [2, Th. 3] In the discrete-time case, IMS (1) is asymptotically stable, if the test matrix $\overline{A} = (\overline{a}_{ij})_{i,j=1,\ldots,n}$ defined below by (4) is Schur stable

$$\overline{a}_{ij} = \sup_{A \in A^I} \{|a_{ij}|\} = \max\{|a_{ij}^-|, |a_{ij}^+|\}, \qquad i,j = 1,\ldots,n. \quad \square \tag{4}$$

It is worth noticing that the test matrix $\overline{A}$ has a particular structure, namely it is essentially nonnegative (i.e., the off-diagonal elements are nonnegative) when defined by (3) and nonnegative (i.e., all the elements are nonnegative) when defined by (4).

Works such as [1]–[3] also considered supplementary hypotheses for the interval matrix A^I (2) to allow the conversion of Lemma 1 into necessary and sufficient conditions for the AS of IMS (1) in Hurwitz [1, Cor. 2.3], [3, Th. 1], or Schur [1, Cor. 1.3], [2, Cor. 1] (time-invariant case) sense. Papers [2], [4], and [5] used the test matrix $\overline{A}$ for referring to the stability degree (margin) of IMS (1) in Hurwitz [4, Cor. 2], [5, Ths. 3.1, 3.4, and 3.5], or Schur [2, Cor. 7] sense.

Despite the importance of all these results, the absence of the converse parts for Lemma 1 i) and ii) has never been examined beyond the simple explanation that $\overline{A}$ might not belong to A^I and/or the formulation of suitable counterexamples. The current note gives a new and complete interpretation of the test matrix $\overline{A}$, as well as of the framework constructed around it for the stability analysis of IMS (1), by using the concepts of *componentwise asymptotic stability* (CWAS) and *componentwise exponential asymptotic stability* (CWEAS). These concepts were introduced and characterized in [6]–[8] as special types of AS derived from the analysis of time-dependent symmetrical rectangular sets, flow-invariant with respect to the state-space trajectories. Papers [6] and [7] focused on continuous-time linear dynamical systems with constant coefficients, whereas [8] extended the investigation area toward continuous-time nonlinear dynamical systems. The study of CWAS and CWEAS for IMSs, in both continuous- and discrete-time cases, was approached by [9] and [10] that dealt with the flow-invariance of IMS dynamics under the same assumption of symmetrical constraints for the state-space trajectories as in [6] and [7].

This note redefines the CWAS and CWEAS concepts in a broader sense, which allows arbitrary componentwise constraints, symmetrical, or nonsymmetrical with respect to the equilibrium point $\{0\}$ of IMS (1). Consequently, the results in [9] and [10] represent particular cases of this new scenario that considerably expands our previous work.

The note is organized as follows. Section II provides a technical background referring to the spectra of (essentially) nonnegative matrices. Section III presents the main results and it is constructed so as to highlight the existence of *bidirectional links* between the properties of the operator $\overline{A}$ [built in accordance with (3) or (4)] and CWAS/CWEAS of IMS (1). Thus, the operator $\overline{A}$ is shown to play a more subtle part in the qualitative analysis of IMS dynamics than disclosed by [1]–[5], since $\overline{A}$ characterizes a special type of AS. This fact is further exploited in Section IV for pointing out the key contributions brought by CWAS/CWEAS in refining the standard concept of AS. Throughout the note, our results and comments on componentwise stability cover both the continuous-time and the discrete-time cases, but only for the former there are given complete proofs, as requesting a more delicate treatment; these proofs apply, *mutatis mutandis*, to the latter.

II. Technical Background

We present some results of general interest, referring to the eigenvalues of the (essentially) nonnegative matrices. These results are used in Section III for the qualitative analysis of componentwise stability of IMSs.

Lemma 2: Let $P = (p_{ij})_{i,j=1,\ldots,n}$ be an essentially nonnegative matrix and denote by $\lambda_i(P), i = 1,\ldots,n$, the eigenvalues of P. **i)** If the diagonal entries of P are nonnegative, then P has a real nonnegative eigenvalue (simple or multiple) denoted by $\lambda_{\max}(P)$, which fulfills the dominance condition $|\lambda_i(P)| \leq \lambda_{\max}(P), i = 1,\ldots,n$. **ii)** If the diagonal entries of P are arbitrary, then P has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(P)$, which fulfills the dominance condition $\mathrm{Re}[\lambda_i(P)] \leq \lambda_{\max}(P), i = 1,\ldots,n$. Moreover, for both cases **i)** and **ii)** $p_{ii} \leq \lambda_{\max}(P), i = 1,\ldots,n$.

Proof: It results from [10, Lemma 2.1], and from [11, Cor. 8.1.20]. □

Lemma 3: Let $Q = (q_{ij})_{i,j=1,\ldots,n}$ be an arbitrary matrix and let us denote by $\overline{Q} = (\overline{q}_{ij})_{i,j=1,\ldots,n}$ the matrix constructed from Q in accordance with (4) or (3), i.e., the bar notation is applied to the trivial case when the interval matrix consists of a single matrix Q. **i)** If $\overline{Q}$ is defined by (4) and $\overline{Q} \leq P$, then $|\lambda_i(Q)| \leq \lambda_{\max}(\overline{Q}) \leq \lambda_{\max}(P)$, $i = 1,\ldots,n$. **ii)** If $\overline{Q}$, is defined by (3) and $\overline{Q} \leq P$, then $\mathrm{Re}[\lambda_i(Q)] \leq \lambda_{\max}(\overline{Q}) \leq \lambda_{\max}(P), i = 1,\ldots,n$.

Proof: i) See [11, Th. 8.1.18]. ii) Consider $s > 0$ such that $q_{ii} + s \geq 0, i = 1,\ldots,n$. Thus, the matrices $\overline{Q} + sI \leq P + sI$ are nonnegative and one can use the results from part i). □

Lemma 4: Let P be an (essentially) nonnegative matrix. **a)** If there exist a positive vector $w \in \mathbf{R}^n, w > 0$ and a constant $p \in \mathbf{R}$ such that $Pw \leq pw$, then $\lambda_{\max}(P) \leq p$. **b)** For any $p \in \mathbf{R}, \lambda_{\max}(P) < p$, there exists a positive vector $w \in \mathbf{R}^n, w > 0$ such that $Pw < pw$.

Proof: See the Appendix . □

III. Main Results

A. Componentwise Asymptotic Stability

Definition 1: Consider two vector functions $h^+(t), h^-(t) : \mathbf{T} \to \mathbf{R}^n$ that have positive components $h_i^+(t), h_i^-(t) > 0, i = 1,\ldots,n$, and $\lim_{t\to\infty} h^+(t) = 0, \lim_{t\to\infty} h^-(t) = 0$. In the continuous-time case ($\mathbf{T} = \mathbf{R}_+$), $h^+(t), h^-(t)$ are also continuously differentiable. IMS (1) is called *componentwise asymptotically stable* with respect to $h^+(t), -h^-(t)$, abbreviated as $\mathrm{CWAS}_{(h^+,-h^-)}$, if

$$\begin{aligned}&\forall t_0, t \in \mathbf{T}, t_0 \leq t : -h_i^-(t_0) \leq x_i(t_0) \leq h_i^+(t_0)\\ &\quad\Rightarrow -h_i^-(t) \leq x_i(t) \leq h_i^+(t), \qquad i = 1,\ldots,n\end{aligned} \tag{5}$$

where $x_i(t), i = 1,\ldots,n$, denote the state variables of IMS (1). □

Remark 1: Condition (5) is equivalent to the fact that the vector functions $-h^-(t), h^+(t) : \mathbf{T} \to \mathbf{R}^n$ define a time-dependent rectangular set, which is flow invariant with respect to the state-space trajectories of IMS (1). □

Definition 2: If $h^+(t) = h^-(t) = h(t)$ in Definition 1, then IMS (1) is called *symmetrically componentwise asymptotically stable* with respect to $h(t)$, abbreviated as CWAS_h. □

Note that the abbreviation CWAS is also used as a noun with the meaning given in Section I.

Theorem 1: IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$ *iff* the following inequalities hold for any $t \in \mathbf{T}$:

$$\begin{aligned}&f_i^+(h^+(t), h^-(t)) \leq (h_i^+(t))'\\ &f_i^-(h^+(t), h^-(t)) \leq (h_i^-(t))' \qquad i = 1,\ldots,n\end{aligned} \tag{6}$$

where $(\ )'$ has the same meaning as in (1) and the vector functions $f^+, f^- : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ are defined by

i) in the continuous-time case

$$f_i^{\pm}(h^+(t), h^-(t)) = a_{ii}^+ h_i^{\pm}(t) + \sum_{j=1, j\neq i}^{n} \max\{a_{ij}^+ h_j^{\pm}(t), -a_{ij}^- h_j^{\mp}(t)\}, \qquad i=1,\ldots,n; \tag{7}$$

ii) in the discrete-time case

$$f_i^{\pm}(h^+(t), h^-(t)) = \sum_{j=1}^{n} \max\{a_{ij}^+ h_j^{\pm}(t), -a_{ij}^- h_j^{\mp}(t)\}, \qquad i=1,\ldots,n. \tag{8}$$

Proof: See the Appendix . □

Remark 2: From Theorem 2, one can easily notice that once IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$, it is also $\mathrm{CWAS}_{(ch^+,-ch^-)}$ and $\mathrm{CWAS}_{(ch^-,-ch^+)}$ for any positive constant $c > 0$. □

Corollary 1: IMS (1) is CWAS_h *iff* the following inequality holds for any $t \in \mathbf{T}$:

$$\overline{A} h(t) \le (h(t))' \tag{9}$$

where the operator $(\ )'$ has the same meaning as in (1) and the matrix $\overline{A}$ is built **i)** in the continuous-time case, according to (3); **ii)** in the discrete-time case, according to (4).

Proof: Inequality (9) is a direct consequence of Theorem 1, for $h_j^+(t) = h_j^-(t) = h_j(t)$ in (7) and (8), respectively. □

Our goal is to demonstrate that the test matrix $\overline{A}$, built from the interval matrix A^I, according to (3) or (4), characterizes CWAS of IMS (1) from the point of view of a qualitative analysis.

Theorem 2: IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$, *iff* **i)** in the continuous-time case, $\overline{A}$ built according to (3) is Hurwitz stable, **ii)** in the discrete-time case, $\overline{A}$ built according to (4) is Schur stable.

Proof: See the Appendix . □

B. Componentwise Exponential Asymptotic Stability

The usage of CWAS with respect to particular vector functions $h^+(t), h^-(t)$ of exponential type

i) for the continuous-time case

$$h^+(t) = d^+ e^{rt}, h^-(t) = d^- e^{rt}, d_i^+ > 0, d_i^- > 0, \qquad i=1,\ldots,n, r<0; \tag{10}$$

ii) for the discrete-time case

$$h^+(t) = d^+ r^t, h^-(t) = d^- r^t, d_i^+ > 0, d_i^- > 0, \qquad i=1,\ldots,n, 0<r<1 \tag{11}$$

yields the following definition.

Definition 3:

i) In the continuous-time case, IMS (1) is called *componentwise exponential asymptotically stable*, abbreviated as CWEAS, if there exist two vectors $d^+, d^- \in \mathbf{R}^n$, with positive components $d_i^+ > 0, d_i^- > 0, i = 1,\ldots,n$, and a constant $r < 0$ such that

$$\forall t_0, t \in \mathbf{T} = \mathbf{R}_+, t_0 \le t : -d_i^- e^{rt_0} \le x_i(t_0) \le d_i^+ e^{rt_0} \Rightarrow -d_i^- e^{rt} \le x_i(t) \le d_i^+ e^{rt}, \qquad i=1,\ldots,n. \tag{12}$$

ii) In the discrete-time case, IMS (1) is called componentwise exponential asymptotically stable, abbreviated as CWEAS, if there exist two vectors $d^+, d^- \in \mathbf{R}^n$, with positive components $d_i^+ > 0, d_i^- > 0, i = 1,\ldots,n$, and a constant $0 < r < 1$ such that

$$\forall t_0, t \in \mathbf{T} = \mathbf{Z}_+, t_0 \le t : -d_i^- r^{t_0} \le x_i(t_0) \le d_i^+ r^{t_0} \Rightarrow -d_i^- r^t \le x_i(t) \le d_i^+ r^t, \qquad i=1,\ldots,n. \tag{13}$$ □

Definition 4: If $d^+ = d^- = d \in \mathbf{R}^n, d_i > 0, i = 1,\ldots,n$, in Definition 3, then IMS (1) is called *symmetrically componentwise exponential asymptotically stable*, abbreviated as symmetrically CWEAS. □

Note that the abbreviation CWEAS is also used as a noun with the meaning given in Section I.

Theorem 3: IMS (1) is CWEAS *iff* the following algebraic inequalities are true.

i) In the continuous-time case (with $d_i^+ > 0,\ d_i^- > 0,\ i = 1,\ldots,n$, and $r < 0$)

$$a_{ii}^+ d_i^{\pm} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \max\{a_{ij}^+ d_j^{\pm}, -a_{ij}^- d_j^{\mp}\} \le r d_i^{\pm}, \qquad i=1,\ldots,n. \tag{14}$$

ii) In the discrete-time case (with $d_i^+ > 0,\ d_i^- > 0, \quad i = 1,\ldots,n$, and $0 < r < 1$)

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \max\{a_{ij}^+ d_j^{\pm}, -a_{ij}^- d_j^{\mp}\} \le r d_i^{\pm}, \qquad i=1,\ldots,n. \tag{15}$$

Proof: It is a consequence of Theorem 1, where the vector functions $h^+(t), h^-(t)$ are chosen as in (10) for the continuous-time case, and (11) for the discrete-time case. □

Remark 3: From Theorem 3, one can easily notice that once IMS (1) is CWEAS with an r and d^+, d^-, it is also CWEAS with the same r and cd^+, cd^-, or cd^-, cd^+ for any positive constant $c > 0$. □

Corollary 2: IMS (1) is symmetrically CWEAS *iff* the following inequalities constructed with the test matrix $\overline{A}$ have solutions

i) in the continuous-time case [with $\overline{A}$ defined by (3)]

$$\overline{A} d \le r d, d_i > 0, \qquad i = 1,\ldots,n, r < 0; \tag{16}$$

ii) in the discrete-time case [with $\overline{A}$ defined by (4)]

$$\overline{A} d \le r d, d_i > 0, \qquad i = 1,\ldots,n, 0 < r < 1. \tag{17}$$

Proof: It results directly from Theorem 3 by taking $d^+ = d^- = d$. □

Theorem 4: IMS (1) is CWEAS *iff* **i)** in the continuous-time case, $\overline{A}$ built according to (3) is Hurwitz stable, **ii)** in the discrete-time case, $\overline{A}$ built according to (4) is Schur stable.

Proof: See the Appendix . □

Remark 4: From Theorems 2 and 4, it results that IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$ *iff* IMS (1) is CWEAS. Actually, this equivalence was expected because of the linearity of the dynamics. □

Remark 5: The proof of Theorem 4 (necessity part) shows that $\lambda_{\max}(\overline{A})$ represents the fastest decreasing rate of the exponential functions defining CWEAS of IMS (1) in the continuous-time case. This property is also valid for the discrete-time case, as resulting from a similar proof. □

IV. CWAS Versus AS in Exploring IMS Dynamics

CWAS allows the individual monitoring of each state variable approaching the equilibrium point and, therefore, it represents a noticeable refinement of the standard concept of AS, where the evolution is characterized in the global terms of a vector norm. Therefore, CWAS of IMS (1) represents only a sufficient condition for the standard AS of IMS (1), but one can identify several classes of IMSs for which this condition is also necessary.

Corollary 3: Assume that the interval matrix A^I (2), corresponding to the continuous-time or discrete-time case, fulfils at least one of the following three conditions, respectively.

i) For the continuous-time case: **i-1)** For all $i, j = 1, \ldots, n, i \neq j$, $|a_{ij}^-| \leq a_{ij}^+$. **i-2)** A^I is either lower- or upper-triangular. **i-3)** Let $V = V_D + V_E$ be an extreme vertex of the hyperrectangle described in $\mathbf{R}^{n\times n}$ by A^I, such that $V_D = \overline{A}_D$ is a diagonal matrix, $|V_E| = \overline{A}_E$ has all the diagonal entries equal to 0, with $\overline{A}_D$, $\overline{A}_E$, defined by $\overline{A} = \overline{A}_D + \overline{A}_E$. (The notation $|\ |$ means the matrix built with the absolute values of the entries.) One of the matrices V_E or $-V_E$ is a Morishima matrix.

ii) For the discrete-time case: **ii-1)** For all $i, j = 1, \ldots, n$, $|a_{ij}^-| \leq a_{ij}^+$ or $|a_{ij}^+| \leq -a_{ij}^-$. **ii-2)** A^I is either lower- or upper-triangular. **ii-3)** Let V be an extreme vertex of the hyperrectangle described in $\mathbf{R}^{n\times n}$ by A^I, such that $|V| = \overline{A}$. (The notation $|\ |$ means the matrix built with the absolute values of the entries.) One of the matrices V or $-V$ is a Morishima matrix.

Then IMS (1) is AS *iff* IMS (1) is CWAS.

Proof: For i-1) and ii-1), $\overline{A}$ belongs to A^I. For i-2) and ii-2), all $A \in A^I$ have real eigenvalues $\lambda_i(A) = a_{ii}, i = 1, \ldots, n$. For i-3) and ii-3), one can use the necessary and sufficient condition for AS of IMS (1) given by [1, Cor. 2.3], and ([1, Cor. 1.3], respectively. □

Within the context of CWAS analysis, the role of the test matrix $\overline{A}$, built from A^I in accordance with (3) or (4), becomes entirely understood in terms of the IMS dynamics. This is due to the fact that our approach is not limited to the algebraic point of view in interpreting A^I, like other works, but also highlights the intrinsic dynamical value of $\overline{A}$. In the symmetrical case of CWAS, the test matrix $\overline{A}$ is just the generator of the semigroup of linear operators $\bar{\Phi}(t_0, t) = e^{\overline{A}(t-t_0)}$, or $\Phi(t_0, t) = A^{(t-t_0)}$ which allows expressing the solutions of inequality (9), in the continuous- or discrete-time case, respectively.

Operator $\overline{A}$ and Stability Tests: Theorem 2 proves that the stability tests based on matrix $\overline{A}$ (3) or (4) actually characterize CWAS of IMS (1), which is a stronger property than the standard AS. In other words, both direct and converse parts become valid for Lemma 1 if its formulation refers to CWAS, instead of AS. Along the same lines, our Corollary 3 offers a deeper explanation for the transformation of Lemma 1 into a necessary and sufficient condition of AS for IMS (1) (because the supplementary hypotheses considered for the interval matrix A^I ensure the equivalence between AS and CWAS). Thus, i-1) in our Corollary 3 covers the hypothesis in [3]; i-3) and ii-3) in our Corollary 3 cover the hypotheses in [1, Cor. 2.3] and [1, Cor. 1.3], respectively; ii-1) and ii-2) in our Corollary 3 cover the hypotheses in [2, Cor. 1] (time-invariant case).

Operator $\overline{A}$ and Gershgorin's Disks: Corollary 2 discloses the whole meaning of the generalized Gershgorin's disks associated with $\overline{A}$ defined by (16) in the continuous-time case and (17) in the discrete-time case, as it clarifies the precise contributions of both vector $d \in \mathbf{R}^n, d > 0$ and scalar $r \in \mathbf{R}$ to the characterization of the state-space trajectories of IMS (1). Unlike Corollary 2, the part played by vector $d \in \mathbf{R}^n, d > 0$ in inequality (16) for defining the time-dependent invariant sets remains hidden for [5, Th. 3.5], where an inequality of form (16) is used only as a sufficient condition for a prescribed AS degree $-r > 0$ of IMS (1).

Operator $\overline{A}$ and Stability Margin/Degree: Theorem 2 allows regarding $|\lambda_{\max}(\overline{A})|$ (in the continuous-time case) or $1/\lambda_{\max}(\overline{A})$ (in the discrete-time case) as the CWAS margin of IMS (1). On the other hand, $|\lambda_{\max}(\overline{A})|$ or $1/\lambda_{\max}(\overline{A})$ represents an upper bound for the AS margin of IMS (1), since Lemma 3 yields: i) in the continuous-time case $\forall A \in A^I : \mathrm{Re}[\lambda_i(A)] \leq \lambda_{\max}(\bar{A}), i = 1, \ldots, n$, and ii) in the discrete-time case $\forall A \in A^I : |\lambda_i(A)| \leq \lambda_{\max}(\bar{A}), i = 1, \ldots, n$. Hence, in the continuous-time case, the sufficient condition in [5, Th. 3.5] must be regarded as a necessary and sufficient condition for a prescribed CWAS degree of IMS (1), because the AS degree is allocated via $\lambda_{\max}(\overline{A})$. In the discrete-time case, the essence of [2, Cor. 7] is actually related to the CWAS margin of IMS (1), since the supplementary hypotheses considered for A^I are of type ii-1) and ii-2) in our Corollary 3 and guarantee the equivalence between CWAS and AS.

V. Conclusion

The usage of CWAS/CWEAS as a refined tool in the investigation of IMS dynamics provides a deeper insight into the role played by the operator $\overline{A}$ in the qualitative analysis, compared to the settings of algebraic nature developed by the works referred to in our note. The key contribution is the equivalence between the stability of $\overline{A}$ and the CWAS of IMS, which is a special type of AS that ensures the flow invariance of certain time-dependent sets with respect to the state-space trajectories. Therefore, those results in literature which address the AS of IMSs via the test matrix $\overline{A}$ can be reformulated as necessary and sufficient conditions with respect to CWAS. Consequently, the stability of $\overline{A}$ becomes a necessary and sufficient condition for the AS of an IMS only when supplementary hypotheses on the interval matrix A^I confer the stronger property of CWAS to that IMS. Moreover, CWAS gives a complete interpretation to the Gershgorin's disks associated with $\overline{A}$, as well as to their exploitation in the analysis of the AS margin/degree, which is actually controlled by its upper bound $\lambda_{\max}(\overline{A})$.

Appendix

A. Proof of Lemma 4

When P is nonnegative, statement a) results from [11, Cor. 8.1.29]. b) For any $p \in \mathbf{R}$, $\lambda_{\max}(P) < p$, there exists an $\varepsilon = \varepsilon(p) > 0$ such that $\lambda_{\max}(P + \varepsilon E) \leq p$, where $E = (e_{ij})_{i,j=1,\ldots,n}$, with $e_{ij} = 1, i, j = 1, \ldots, n$. Thus, for the Perron eigenvector $w \in \mathbf{R}^n, w > 0$, of the positive matrix $P + \varepsilon E > 0$ we can write $Pw < (P + \varepsilon E)w = \lambda_{\max}(P + \varepsilon E)w \leq pw$. When P is essentially nonnegative, we construct the nonnegative matrix $sI + P$ with $s + p_{ii} \geq 0$ and conduct the proof along the same lines as before. Note that when P is irreducible, the Perron-Frobenius eigenvector $w \in \mathbf{R}^n, w > 0$ fulfills $Pw = \lambda_{\max}(P)w$, and statements a) and b) can be replaced by "there exist a positive vector $w \in \mathbf{R}^n$, $w > 0$ and $p \in \mathbf{R}$ such that $Pw \leq pw$ *iff* $\lambda_{\max}(P) \leq p$." □

B. Proof of Theorem 1

In both continuous- and discrete-time cases, the $\mathrm{CWAS}_{(h^+,-h^-)}$ condition is equivalent to the flow invariance of the time-dependent rectangular set

$$H(t) = \left[-h_1^-(t), h_1^+(t)\right] \times \cdots \times \left[-h_n^-(t), h_n^+(t)\right] \quad \text{(A1)}$$

with respect to IMS (1), where the vector functions $h^+(t), h^-(t)$ fulfill the requirements in Definition 1. In the discrete-time case, the exploitation of $H(t)$ is straightforward. In the continuous-time case, the flow invariance is approached in terms of [12, Ch. II, Lemma 4.2], i.e., we can write

$$a_{ii}h_i^+(t) + \sum_{j=1, j\neq i}^{n} a_{ij}x_j(t) \leq (h_i^+(t))'$$
$$-(h_i^-(t))' \leq -a_{ii}h_i^-(t) + \sum_{j=1, j\neq i}^{n} a_{ij}x_j(t)$$
$$\forall t \geq 0, \qquad i = 1, \ldots, n \quad \text{(A2)}$$

for $x_j(t) \in [-h_j^-(t), h_j^+(t)]$, $j = 1, \ldots, i-1, i+1, \ldots, n$, and for $a_{ij} \in [a_{ij}^-, a_{ij}^+]$, $i = 1, \ldots, n$. Thus, an equivalent form for (A2) is

$$a_{ii}^+ h_i^+(t) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \max_{\substack{a_{ij}^- \leq a_{ij} \leq a_{ij}^+ \\ -h_j^-(t) \leq x_j(t) \leq h_j^+(t)}} (a_{ij} x_j(t)) \leq (h_i^+(t))' \quad \forall t \geq 0, \quad i = 1, \ldots, n \tag{A3a}$$

$$-(h_i^-(t))' \leq -a_{ii}^+ h_i^-(t) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \min_{\substack{a_{ij}^- \leq a_{ij} \leq a_{ij}^+ \\ -h_j^-(t) \leq x_j(t) \leq h_j^+(t)}} (a_{ij} x_j(t)) \quad \forall t \geq 0, \quad i = 1, \ldots, n \tag{A3b}$$

where

$$\max_{\substack{a_{ij}^- \leq a_{ij} \leq a_{ij}^+ \\ -h_j^-(t) \leq x_j(t) \leq h_j^+(t)}} (a_{ij} x_j(t)) = \max\{a_{ij}^+ h_j^+(t), -a_{ij}^- h_j^-(t)\} \quad \forall t \geq 0 \tag{A4a}$$

$$\min_{\substack{a_{ij}^- \leq a_{ij} \leq a_{ij}^+ \\ -h_j^-(t) \leq x_j(t) \leq h_j^+(t)}} (a_{ij} x_j(t)) = \min\{a_{ij}^+(-h_{ij}^-(t)), a_{ij}^- h_j^+(t)\} = -\max\{a_{ij}^+ h_j^-(t), -a_{ij}^- h_j^+(t)\} \quad \forall t \geq 0. \tag{A4b}$$

By replacing (A4) in (A3), we get inequalities (6), with $f^+, f^- : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ given by (7). □

C. *Proof of Theorem 2*

(For the continuous-time case) First, we prove that $\overline{A}$ Hurwitz stable is a necessary and sufficient condition for IMS (1) to be symmetrically CWAS, in the sense of Definition 2. *Necessity:* Condition (9) is equivalent to $\forall t, t_0 \in \mathbf{R}_+, t_0 \leq t : e^{\overline{A}(t-t_0)} h(t_0) \leq h(t)$, and $\lim_{t\to\infty} h(t) = 0$ means the existence of $t > t_0$ such that $h(t) \leq \varphi h(t_0)$ with $0 < \varphi < 1$, i.e., $e^{\overline{A}(t-t_0)} h(t_0) \leq \varphi h(t_0)$ with $h(t_0) > 0$, $0 < \varphi < 1$. Since $e^{\overline{A}(t-t_0)}$ is a nonnegative matrix, Lemma 4a) yields $\lambda_{\max}(e^{\overline{A}(t-t_0)}) \leq \varphi < 1$, i.e., $\lambda_{\max}(\overline{A}) < 0$. *Sufficiency:* If $\overline{A}$ is Hurwitz stable, then IMS (1) is $\mathrm{CWAS}_{(h,-h)}$ for any $h(t) = e^{\overline{A}t} h(0) + \int_0^t e^{\overline{A}(t-\tau)} u(\tau) d\tau$ with $h(0) > 0$ and adequate $u(\tau) \geq 0$, $\tau \geq 0$ such that $\lim_{t\to\infty} \int_0^t e^{\overline{A}(t-\tau)} u(\tau) d\tau = 0$.

Now, we can give the proof for the broader sense of CWAS, as stated by Definition 1. *Necessity:* If IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$, then, according to Theorem 1, the summation of (6), for each $i = 1, \ldots, n$, yields the inequalities

$$\sum_{j=1}^{n} f_{ij}(t) \leq (h_i(t))', \qquad i = 1, \ldots, n \tag{A5}$$

where

$$f_{ii}(t) = a_{ii}^+ (h_i^+(t) + h_i^-(t))$$
$$\underset{i\neq j}{f_{ij}(t)} = \max\{a_{ij}^+ h_j^+(t), -a_{ij}^- h_{ij}^-(t)\} + \max\{a_{ij}^+ h_j^-(t), -a_{ij}^- h_{ij}^+(t)\} \tag{A6a}$$
$$h_i(t) = h_i^+(t) + h_i^-(t). \tag{A6b}$$

Using the notations introduced in (A6) and the entries $\overline{a}_{ij}$ of $\overline{A}$, defined by (3), let us show that

$$\overline{a}_{ij} h_j(t) \leq f_{ij}(t), \qquad i, j = 1, \ldots, n. \tag{A7}$$

If $i = j$, inequalities (A7) are satisfied as equalities. If $i \neq j$, we have cases **a)**, **b)**, and **c)** detailed as follows. Cases **a)** $0 \leq a_{ij}^- \leq a_{ij}^+ = \overline{a}_{ij}$ and **b)** $-\overline{a}_{ij} = a_{ij}^- \leq a_{ij}^+ \leq 0$ imply $f_{ij}(t) = \overline{a}_{ij} h_j(t)$. Case **c)** $a_{ij}^- < 0 < a_{ij}^+$ implies $f_{ij}(t) \geq \overline{a}_{ij} h_j(t)$, for both situations **c1)** $|a_{ij}^-| \leq a_{ij}^+ = \overline{a}_{ij}$, **c2)** $\overline{a}_{ij} = |a_{ij}^-| > a_{ij}^+$, completing the proof of inequalities (A7). From (A5) and (A7) we conclude that inequality (9) is true for any $t \geq 0$, with $h(t)$ a positive and continuously differentiable vector function, meeting the condition $\lim_{t\to\infty} h(t) = 0$. Thus, from Corollary 1, IMS (1) results symmetrically CWAS, and, in accordance with the first part of the current proof, the test matrix $\overline{A}$ is Hurwitz stable. *Sufficiency:* If the test matrix $\overline{A}$ is Hurwitz stable, let us show that two different vector functions $h^+(t), h^-(t)$ can be found such that IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$. Take a positive constant $s > 0$, such that $s \geq |\overline{a}_{ii}|$, $i = 1, \ldots, n$, and $s > |\lambda_{\max}(\overline{A})|$, where the notation $\lambda_{\max}()$ has the meaning introduced by Lemma 2 and consider an arbitrary positive $v < |\lambda_{\max}(\overline{A})|/(s - |\lambda_{\max}(\overline{A})|)$. As resulting from Lemma 2, in most cases $\max_{i=1,\ldots,n}\{|a_{ii}|\} > |\lambda_{\max}(\overline{A})|$, when we can choose $s = \max_{i=1,\ldots,n}\{|a_{ii}|\}$. The essentially nonnegative matrix, built with the elements of the test matrix $\overline{A}$

$$B = (b_{ij})_{i,j=1,\ldots,n}, b_{ii} = \overline{a}_{ii}, b_{ij} = (1+v)\overline{a}_{ij}, i \neq j \tag{A8}$$

is Hurwitz stable, because, from the matrix inequality $B = (1+v)\overline{A} - v\mathrm{diag}\{\overline{a}_{11}, \ldots, \overline{a}_{nn}\} \leq (1+v)\overline{A} + vsI$, we obtain the eigenvalue inequality $\lambda_{\max}(B) \leq \lambda_{\max}((1+v)\overline{A} + vsI) = (1+v)\lambda_{\max}(\overline{A}) + vs < 0$ (from Lemma 3). The Hurwitz stability of B guarantees the existence of a positive vector function $h(t) > 0$, continuously differentiable, with $\lim_{t\to\infty} h(t) = 0$, which satisfies the differential inequality $Bh(t) \leq (h(t))'$, e.g., $h(t) = e^{Bt} h(0), h(0) > 0$. Consider two sets of arbitrary positive constants $c_i^+ > 0, c_i^- > 0, i = 1, \ldots, n$, such that

$$\frac{c_j^+}{c_i^+}, \frac{c_j^-}{c_i^+}, \frac{c_j^-}{c_i^-}, \frac{c_j^+}{c_i^-} < 1 + v \qquad \text{for all } i \neq j. \tag{A9}$$

Define the vector functions $h^+(t), h^-(t)$, by $h_i^+(t) = c_i^+ h_i(t)$, $h_i^-(t) = c_i^- h_i(t)$, $i = 1, \ldots, n$, that are used to evaluate the right-hand side of (7)

$$\begin{aligned} & f_i^{\pm}(h^+(t), h^-(t)) \\ &= a_{ii}^+ c_i^{\pm} h_i(t) + \sum_{j=1, j\neq i}^{n} \max\left\{a_{ij}^+ c_j^{\pm} h_j(t), -a_{ij}^- c_j^{\mp} h_j(t)\right\} = \\ &= c_i^{\pm}\left[a_{ii}^+ h_i(t) + \sum_{j=1, j\neq i}^{n} \max\left\{a_{ij}^+ \frac{c_j^{\pm}}{c_i^{\pm}}, -a_{ij}^- \frac{c_j^{\mp}}{c_i^{\pm}}\right\} h_j(t)\right] \\ &\leq c_i^{\pm}\left[\overline{a}_{ii} h_i(t) + \sum_{j=1, j\neq i}^{n} (1+v)\overline{a}_{ij} h_j(t)\right] = \\ &= c_i^{\pm}\left[\sum_{j=1}^{n} b_{ij} h_j(t)\right] \leq c_i^{\pm}(h_i(t))' = (h_i^{\pm}(t))', \qquad i = 1, \ldots, n. \end{aligned} \tag{A10}$$

Thus, according to Theorem 1, IMS (1) is $\mathrm{CWAS}_{(h^+,-h^-)}$, because inequalities (6) are fulfilled via inequalities (A10), and the vector functions $h^+(t) \neq h^-(t)$ meet the conditions in Definition 1. □

TABLE I
DISCUSSION PROVIDING THE VALUES OF THE COEFFICIENTS $\tilde{a}_{ij}^{11}, \tilde{a}_{ij}^{12}, \tilde{a}_{ij}^{21}, \tilde{a}_{ij}^{22}$ IN (A11)

$i,j=1,\ldots,n$	Distinct cases for $a_{ij}^- \le a_{ij}^+$		$\tilde{a}_{ij}^{11}$	$\tilde{a}_{ij}^{12}$	$\tilde{a}_{ij}^{21}$	$\tilde{a}_{ij}^{22}$
$i=j$	$a_{ii}^+ = \overline{a}_{ii}$		$\overline{a}_{ii}$	0	0	$\overline{a}_{ii}$
$i \ne j$	$0 \le a_{ij}^- \le a_{ij}^+ = \overline{a}_{ij}$		$\overline{a}_{ij}$	0	0	$\overline{a}_{ij}$
	$-\overline{a}_{ij} = a_{ij}^- \le a_{ij}^+ \le 0$		0	$\overline{a}_{ij}$	$\overline{a}_{ij}$	0
	$a_{ij}^- < 0 < a_{ij}^+$	$\lvert a_{ij}^- \rvert < a_{ij}^+ = \overline{a}_{ij}$	$\overline{a}_{ij}$	0	0	$\overline{a}_{ij}$
		$a_{ij}^+ < \lvert a_{ij}^- \rvert = \overline{a}_{ij}$	0	$\overline{a}_{ij}$	$\overline{a}_{ij}$	0
		$\lvert a_{ij}^- \rvert = a_{ij}^+ = \overline{a}_{ij}$	$\overline{a}_{ij}$	0	0	$\overline{a}_{ij}$
			OR			
			0	$\overline{a}_{ij}$	$\overline{a}_{ij}$	0

D. Proof of Theorem 4

(For the continuous-time case) *Necessity:* Starting from inequalities (14) that are equivalent to the CWEAS of IMS (1), we can write

$$\sum_{j=1}^{n}\tilde{a}_{ij}^{11} d_j^+ + \sum_{j=1}^{n}\tilde{a}_{ij}^{12} d_j^- \le a_{ii}^+ d_i^+ + \sum_{j=1,j\ne i}^{n} \max\{a_{ij}^+ d_j^+, -a_{ij}^- d_j^-\} \le r d_i^+, \qquad i=1,\ldots,n \tag{A11a}$$

$$\sum_{j=1}^{n}\tilde{a}_{ij}^{21} d_j^+ + \sum_{j=1}^{n}\tilde{a}_{ij}^{22} d_j^- \le a_{ii}^+ d_i^- + \sum_{j=1,j\ne i}^{n} \max\{a_{ij}^+ d_j^-, -a_{ij}^- d_j^+\} \le r d_i^-, \qquad i=1,\ldots,n \tag{A11b}$$

where the values of the coefficients $\tilde{a}_{ij}^{11}, \tilde{a}_{ij}^{12}, \tilde{a}_{ij}^{21}, \tilde{a}_{ij}^{22}, i,j=1,\ldots,n$, are given by the discussion in Table I, as expressed in terms of the entries $\overline{a}_{ij}$ of the test matrix $\overline{A}$.

Thus, if IMS (1) is CWEAS, then $\tilde{a}_{ij}^{22} = \tilde{a}_{ij}^{11}, \tilde{a}_{ij}^{21} = \tilde{a}_{ij}^{12}$ and the following inequalities are true:

$$\tilde{A}\begin{bmatrix} d^+ \\ d^- \end{bmatrix} = \begin{bmatrix} \tilde{A}^{11} & \tilde{A}^{12} \\ \tilde{A}^{12} & \tilde{A}^{11} \end{bmatrix}\begin{bmatrix} d^+ \\ d^- \end{bmatrix} \le r \begin{bmatrix} d^+ \\ d^- \end{bmatrix}$$
$$d^+ > 0, d^- > 0, r < 0 \tag{A12}$$

which, according to Lemma 4a), require $\lambda_{\max}(\tilde{A}) \le r < 0$. On the other hand, for $\tilde{A}$ we can write

$$\tilde{A} = \begin{bmatrix} \tilde{A}^{11} & \tilde{A}^{12} \\ \tilde{A}^{12} & \tilde{A}^{11} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \cdot \begin{bmatrix} \tilde{A}^{11} + \tilde{A}^{12} & 0 \\ 0 & \tilde{A}^{11} - \tilde{A}^{12} \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$
$$\begin{bmatrix} I & I \\ I & -I \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \tag{A13}$$

where $\tilde{A}^{11} + \tilde{A}^{12} = \overline{A}$ and $\overline{\tilde{A}^{11} - \tilde{A}^{12}} = \overline{A}$, the bar notation having the same meaning as in (3). Since Lemma 3 ensures $\mathrm{Re}[\lambda_i(\tilde{A}^{11} - \tilde{A}^{12})] \le \lambda_{\max}(\overline{A}), i=1,\ldots,n$, we have $\lambda_{\max}(\overline{A}) = \lambda_{\max}(\tilde{A}) \le r < 0$. *Sufficiency:* If $\overline{A}$ is Hurwitz stable, then the matrix B built according to (A8) is also Hurwitz stable as shown in the proof of Theorem 2. Hence, in accordance with Lemma 4b), for any $r \in \mathbf{R}$, $\lambda_{\max}(B) < r < 0$, we can find a positive vector $d = [d_1 \ldots d_n]^T > 0$ ($[\,]^T$ denoting the transposition) such that the inequality $Bd < rd$ is fulfilled. By using the constants $c_i^+ > 0, c_i^- > 0, i=1,\ldots,n$, meeting conditions (A9), define the positive vectors $d_i^+ = c_i^+ d_i, d_i^- = c_i^- d_i$, $i=1,\ldots,n$, for which we have

$$\begin{aligned} a_{ii}^+ d_i^{\pm} &+ \sum_{j=1,j\ne i}^{n} \max\{a_{ij}^+ d_j^{\pm}, -a_{ij}^- d_j^{\mp}\} \\ &= a_{ii}^+ c_i^{\pm} d_i + \sum_{j=1,j\ne i}^{n} \max\{a_{ij}^+ c_j^{\pm} d_j, -a_{ij}^- c_j^{\mp} d_j\} \\ &= c_i^{\pm}\left[a_{ii}^+ d_i + \sum_{j=1,j\ne i}^{n} \max\left\{a_{ij}^+ \frac{c_j^{\pm}}{c_i^{\pm}}, -a_{ij}^- \frac{c_j^{\mp}}{c_i^{\pm}}\right\} d_j\right] \\ &\le c_i^{\pm}\left[\overline{a}_{ii} d_i + \sum_{j=1,j\ne i}^{n} (1+v)\overline{a}_{ij} d_j\right] = \\ &= c_i^{\pm}\left(\sum_{j=1}^{n} b_{ij} d_j\right) < c_i^{\pm} r d_i = r d_i^{\pm}, \qquad i=1,\ldots,n. \end{aligned} \tag{A14}$$

The proof is completed, because inequalities (14) are satisfied. □

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